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طريقة مباشرة لحل بعض مسائل القيم الابتدائية ذات المعادلة التفاضلية العادية

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□ ملخّص □

نقدم في هذا العمل، طريقة لحل بعض مسائل القيم الابتدائية ذات المعادلات التفاضلية العادية الخطية وغير الخطية من المرتبة الأولى وحتى المرتبة الثالثة. يتم تطبيق الطريقة مباشرة لإيجاد الحلول للمسائل المذكورة دون تخفيض المرتبتين الثانية والثالثة إلى جملة معادلات تفاضلية من المرتبة الأولى كما هو معتاد في الطرق التقليدية. تمت دراسة التناسق والاستقرار المطلق وإيجاد الأخطاء المقتطعة الشاملة للطريقة عندما تم تطبيقها على المسائل المذكورة. لإثبات صحة النتائج النظرية قمنا باختبار الطريقة المقترحة بحل خمس مسائل مختلفة في المعادلات التفاضلية من المراتب الأولى والثانية والثالثة، حيث تشير المقارنات لنتائجنا مع نتائج الآخرين إلى أفضلية الطريقة المقترحة من حيث الدقة والفعالية.

االكلمات المفتاحية: معادلات تفاضلية عادية، مسائل القيم الابتدائية ، التناسق ، الخطأ المقتطع الشامل، الاستقرار المطلق.

* أستاذ مساعد في قسم الرياضيات بكلية العلوم– جامعة تشرين– اللاذقية– سورية

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Direct Method to Solve Some Initial-Value Problems in Ordinary Differential Equations

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□ABSTRACT □

In this paper, direct method is presented to solve some initial values problems in linear and nonlinear ordinary differential equations from the first up to the third order.
The proposed method is applied directly to finding solutions to differential equations without reducing the second and third order to systems of first order equations as other methods. Therefore, consistency and absolute stability are studied, and then global truncations errors are found for the method applied to the mentioned problems. To demonstrate the validity of the theoretical results, we tested the proposed method by solving five problems in differential equations of the first, second and third order, where comparisons of our results with the results of others indicate the efficiency of the proposed method in terms of accuracy and effectiveness.

Keywords: Ordinary Differential Equations, Initial-Value Problems, Consistency, Global Truncation Error, Absolute stability.

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1. Introduction

Ordinary differential equations (ODEs) and their solutions play a major role in mechanics, physics, chemistry, economics and astronomy etc. Sometimes, it is very difficult to find an analytical solution to a given ODEs. For this reasons it is important to search for alternative techniques to find such solutions.

This paper describes a direct method for solving the general initial value problems (IVPs) in ordinary differential equations, viz.

$$y^{(n)}(x) = f(x, y, y', ..., y^{(n-1)}), \ x \in [a,b] \ , n=1,2,3, \qquad (1)$$

$$y^{(k)}(a) = y_0^{(k)}, \ k=0,1,\ldots,n-1. \qquad (2)$$

The problem (1)-(2) for n=2, occurs frequently in Celestial Mechanics and, for example in mechanical problems, without dissipation. Theoretical solutions of such problems are normally highly oscillatory.

Recently, many methods have been introduced to solve the problem (1)-(2) for n=1,2,3, such that:

Mahmoud and Osman [8] studied a class of spline-collocation methods for solving second-order IVPs; their method was stable and convergent from the fifth order. Eskandari and Dahaghin [6] produced a special general linear multi-step method for special second-order differential equations, in addition to checking stability. Yap et al [14] proposed the block hybrid collocation method with two off-step points for - order IVPs. Their method is derived via interpolation and collocation of the basic polynomial. The stability properties of the block method are investigated. Agboola et al. [2] study a simple and Taylor series-based method known as differential transformation method to solve third-order IVPs. Ramos et al [12] found the construction of a family of explicit schemes for the numerical solution of IVPs of ordinary differential equations .The one-parameter family is constructed by considering a suitable rational approximation to the theoretical solution, their method is A-stable and second-order convergence.

Duromola and Momoh [5] developed a hybrid method with block extension to solve third-order IVPs. The derived method was tested for consistency, zero stability, convergence and absolute stability. Abdelrahim [1] was thought numerical hybrid block method together with shooting technique to be an appropriate method for solving third-order BVPs. Ahmoud et al. [**9**] presented a numerical spline algorithm for the Falkner–Skan problem over a semi-infinite interval. Their algorithm is based on change of variable from interval $[0,\infty[$ to [0,1], then the FSE is transformed into first initial value problem (IVP) and second IVP for improving convergence. Sahu and Jena [13] introduced a novel operational matrix method for solving, This method is based on the frame of linear cardinal B-spline. Fairuz and. Majid [7] presented a class of rational methods of second to fourth order of accuracy. The absolute stability and comparisons with others are considered. Oje and Majuk in [10] considered rational Runge-Kutta schemes to solve second-order IVPs, then stability and convergence are studied.

2. Importance of Research and its Objectives

It is well known that most phenomena in nature and realistic applications can be simulated by models of ordinary differential equations, but unfortunately the greatest part of these equations cannot be solved by classical methods, for this reason we aim to present effective and stable method to obtain approximate solutions for such IVPs in ordinary differential equations(1)-(2).

3. Methodology

Ordinary differential equations are closely related to several sciences, including numerical analysis, linear algebra, mathematical analysis, and computer science. Therefore, the proposed method depends on creating approximating solution to the problem by polynomial of a certain degree, then studying stability, consistency, and determining the formula for global truncated error.

4. Results and discussion

Firstly, we begin to provide the following definition.

Definition 1. A one-step difference-equation method with local truncation error $\bar{\tau}_i\ (h)$ at the *i*th step is said to be **consistent** with the differential equation and boundary conditions if

$\lim_{h\to 0} \max_{1\le i\le N} |\bar{\tau}_i|=0.$

4.1 Description of the SC Methods

Let us consider the initial value problem (1)-(2) and assume that $f \in C^5([a,b]\times\Re^n)$, and that it satisfies the Lipschitiz condition:

$| f(x, y_1, y_2,..., y_n) - f(x, y_1^*, y_2^*,..., y_n^*) |\le L\sum_{i=1}^{n}| y_i - y_i^* |$, in $[a,b]\times\Re^n$.

The approximate solution $P(x)$ is constructed as follows:

Subdivide $[a,b]$ into sub intervals of equal length $h=(b-a)/N$, with grid points $x_i = a + i\,h$, $i=0,1,\ldots,N-1$, and let the component of $P(x)$ on $\mathbf{I}_i = [x_i, x_{i+1}]$ are given as [**8**]:

$$\begin{aligned} \boldsymbol{P(x)} &= \bar{t}^3(6t^2+3t+1)y_i + \bar{t}^3(3t^2+t)hy'_i + (\tfrac{1}{2}t^2)\bar{t}^3h^2y''_i \\ &\quad + t^3(6\bar{t}^2+3\bar{t}+1)y_{i+1} - t^3(3\bar{t}^2+\bar{t})hy'_{i+1} + t^3(\tfrac{1}{2}\bar{t}^2)h^2y''_{i+1} \end{aligned} \tag{3}$$

where $\boldsymbol{t = (x - x_i)/h \in [0,1],\ \bar{t} = 1 - t}$.

We need the derivatives of $\boldsymbol{P}(x)$ with respect to x up to third-order

$$\begin{aligned}\boldsymbol{P'(x)} &= -30\bar{t}^2 t^2 y_i / h + \bar{t}^2(1+2t-15t^2)y'_i + \bar{t}^2(t-\tfrac{5}{2}t^2)hy''_i + \\ &\quad 30\,t^2\,\bar{t}^2\, y_{i+1}/h + t^2(1+2\bar{t}-15\bar{t}^2)y'_{i+1} - t^2(\bar{t}-\tfrac{5}{2}\bar{t}^2)hy''_{i+1}\end{aligned} \tag{4}$$

$$\begin{aligned}\boldsymbol{P''(x)} &= [\bar{t}(120t^2-60t)\,y_i + \bar{t}(60t^2-36t)hy'_i + \bar{t}(10t^2-8t+1)h^2y''_i + \\ &\quad t(120\bar{t}^2-60\bar{t})y_{i+1} + t(36\bar{t}-60\bar{t}^2)hy'_{i+1} + t(10\bar{t}^2-8\bar{t}+1)h^2y''_{i+1}]/h^2\end{aligned} \tag{5}$$

$$\begin{aligned}\boldsymbol{P'''(x)} &= [(360t\,\bar{t}-60)\,y_i + (192t-180t^2-36)hy'_i + (36t-30t^2-9)h^2y''_i + \\ &\quad (60-360\,t\,\bar{t})y_{i+1} + (192\bar{t}-180\bar{t}^2-36)hy'_{i+1} + (30\bar{t}^2-36\bar{t}+9)h^2y''_{i+1}]/h^3\end{aligned} \tag{6}$$

The polynomial $P(x)$ and their derivatives up to third-order with three collocation points are applied to (1)-(2) as followed:

$$\begin{aligned}&P^{(n)}(x_{i+c_1}) = f[x_{i+c_1}, P(x_{i+c_1}), P'(x_{i+c_1}),\dots,P^{(n-1)}(x_{i+c_1})]\ ,\\ &P^{(n)}(x_{i+c_2}) = f[x_{i+c_2}, P(x_{i+c_2}), P'(x_{i-1+c_2}),\dots,P^{(n-1)}(x_{i+c_2})],\\ &P^{(n)}(x_{i+1}) = f[x_{i+1}, P(x_{i+1}), P'(x_{i+1}),\dots,P^{(n-1)}(x_{i+1})],\quad n=1,2,3;\\ &\qquad\qquad i=0,1,\dots,N-1\end{aligned} \tag{7}$$

in each subinterval I_i, where collocation points are given as

$$x_{i+c_1} = x_i + c_1h,\ \ x_{i+c_2} = x_i + c_2h,\ \ x_{i+1} = x_i + h, \tag{8}$$

with the collocation parameters :

$$c_1 = 4/5,\ \ c_2 = 11/12 \tag{9}$$

4.2 Convergence of the proposed method for the first order (n=1)

Putting n=1 in relations (7)-(9) , we get

$$P'(x_{i+c_j}) = f(x_{i+c_j}, P(x_{i+c_j}))\ \ ,\ j=1,2, \tag{10}$$

$$P'(x_{i+1}) = P'_{i+1} = f(x_{i+1}, P(x_{i+1})),\ i=0,1,\dots,\mathrm{N-1} \tag{11}$$

and taking into account substitution in the values $\boldsymbol{P'_i} = \boldsymbol{f_i}$, $\boldsymbol{P'_{i+1}} = \boldsymbol{f_{i-1}}$, and using the collocation parameters (9), we can write (10)-(11) as follows:

$$\begin{bmatrix}\frac{96}{125} & -\frac{8}{125}\\ \frac{605}{3456} & -\frac{2299}{41472}\end{bmatrix}\begin{bmatrix}P_{i+1}\\ h^2P''_{i+1}\end{bmatrix} = \begin{bmatrix}\frac{96}{125} & \frac{4}{125}\\ \frac{605}{3456} & \frac{341}{41472}\end{bmatrix}\begin{bmatrix}P_i\\ h^2P''_i\end{bmatrix} + h\begin{bmatrix}\frac{7}{25} & 1 & 0 & -\frac{64}{125}\\ \frac{469}{6912} & 0 & 1 & -\frac{2057}{2304}\end{bmatrix}\begin{bmatrix}f_i\\ f_{i+c_1}\\ f_{i+c_2}\\ f_{i+1}\end{bmatrix}$$

By solving this system we have

$$\begin{bmatrix}P_{i+1}\\ h^2P''_{i+1}\end{bmatrix} = \begin{bmatrix}1 & \frac{7}{176}\\ 0 & -\frac{1}{44}\end{bmatrix}\begin{bmatrix}P_i\\ h^2P''_i\end{bmatrix} + h\begin{bmatrix}\frac{8279}{23232} & \frac{2375}{1344} & -\frac{1728}{847} & \frac{11}{12}\\ -\frac{191}{1936} & \frac{625}{112} & -\frac{20736}{847} & 19\end{bmatrix}\begin{bmatrix}f_i\\ f_{i+c_1}\\ f_{i+c_2}\\ f_{i+1}\end{bmatrix} \tag{12}$$

Notice that the matrix of the system (12) nonsingular because

$$\begin{vmatrix} 1 & \frac{7}{176} \\ 0 & -\frac{1}{44} \end{vmatrix} = -\frac{1}{44} \neq 0,$$

and that the its eigenvalues lie inside to the unit disc.

We obtain from the system (12) that **local truncation error**:

$$\bar{\tau}_i = \begin{bmatrix} y_{i+1} \\ h^2 y''_{i+1} \end{bmatrix} - \begin{bmatrix} 1 & \frac{7}{176} \\ 0 & -\frac{1}{44} \end{bmatrix} \begin{bmatrix} y_i \\ h^2 y''_i \end{bmatrix} - h \begin{bmatrix} \frac{8279}{23232} & \frac{2375}{1344} & -\frac{1728}{847} & \frac{11}{12} \\ -\frac{191}{1936} & \frac{625}{112} & -\frac{20736}{847} & 19 \end{bmatrix} \begin{bmatrix} y'(x_i) \\ y'(x_{i+c_1}) \\ y'(x_{i+c_2}) \\ y'(x_{i+1}) \end{bmatrix} \quad (13)$$

Using Taylor's expansions of y, y', y'' and substituting in (13), one can estimate the local truncation error as follows:

$$\bar{\tau}_i = h^6 \begin{bmatrix} \frac{31}{432000} \\ -\frac{1}{7200} \end{bmatrix} f^{(6)}(x_i) + O(h^7), i = 1,2,\dots,\mathrm{N}$$

Note that local discretization error $\bar{\tau}_i$ of our method is from order six. For that using the max norm, we get global truncation error:

$||\bar{\tau}_i|| = \frac{1}{7200} \mathrm{c}\, h^5;\ c := \max |f^{(6)}(x_i)|, for\ x_i \in [\mathrm{a,b}]$

Hence, our method applied to first order IVPs is consistent of order five.

4.3 Convergence of the proposed method for the second order (n=2)

Putting n=2 in relations (7)-(9) , we get

$$P''(x_{i+c_j}) = f(x_{i+c_j}, P(x_{i+c_j}), P'(x_{i+c_j})) \ , j = 1,2 \quad (14)$$

$$\boldsymbol{P''_{i+1} = f(x_{i+1}, P(x_{i+1}), P'(x_{i+1})), i = 0,1,\dots,N-1} \quad (15)$$

Substituting in the values $\boldsymbol{P''_i} = \boldsymbol{f_i}$, $\boldsymbol{P''_{i+1}} = \boldsymbol{f_{i-1}}$, and using the collocation parameters (9) the system (13)-(14) are obtained as follows:

$$\begin{bmatrix} -\frac{144}{25} & \frac{96}{25} \\ -\frac{275}{72} & \frac{341}{144} \end{bmatrix} \begin{bmatrix} P_{i+1} \\ hP'_{i+1} \end{bmatrix} = \begin{bmatrix} -\frac{144}{25} & -\frac{48}{25} \\ -\frac{275}{72} & -\frac{209}{144} \end{bmatrix} \begin{bmatrix} P_i \\ hP'_i \end{bmatrix} + h^2 \begin{bmatrix} -\frac{1}{5} & 1 & 0 & \frac{4}{25} \\ -\frac{149}{864} & 0 & 1 & -\frac{319}{864} \end{bmatrix} \begin{bmatrix} f_i \\ f_{i+c_1} \\ f_{i+c_2} \\ f_{i+1} \end{bmatrix}$$

By solving this system we have

$$\begin{bmatrix} P_{i+1} \\ hP'_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_i \\ hP'_i \end{bmatrix} + h^2 \begin{bmatrix} \frac{97}{528} & \frac{775}{336} & -\frac{288}{77} & \frac{7}{4} \\ \frac{59}{264} & \frac{625}{168} & -\frac{432}{77} & \frac{8}{3} \end{bmatrix} \begin{bmatrix} f_i \\ f_{i+c_1} \\ f_{i+c_2} \\ f_{i+1} \end{bmatrix}$$

(16)

Notice that the matrix of the system (16) nonsingular because

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

and that the its eigenvalues lie onside to the unit disc.

We obtain from the system (12) that the **local truncation error**:

$$\bar{\tau}_i = \begin{bmatrix} y_{i+1} \\ hy'_{i+1} \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_i \\ hy'_i \end{bmatrix} - h^2 \begin{bmatrix} \frac{97}{528} & \frac{775}{336} & -\frac{288}{77} & \frac{7}{4} \\ \frac{59}{264} & \frac{625}{168} & -\frac{432}{77} & \frac{8}{3} \end{bmatrix} \begin{bmatrix} y''(x_i) \\ y''(x_{i+c_1}) \\ y''(x_{i+c_2}) \\ y''(x_{i+1}) \end{bmatrix} \qquad (17)$$

Using Taylor's expansions of y, y', y'' and substituting in (17), one can estimate the local truncation error as follows:

$$\bar{\tau}_i = h^6 \begin{bmatrix} \frac{37}{43200} \\ \frac{7}{5760} \end{bmatrix} f^{(6)}(x_i) + O(h^7), i = 1,2,\dots,\mathrm{N}$$

Note that local discretization error $\bar{\tau}_i$ of this method is from order six, for this using max norm we get global truncation error:

$||\bar{\tau}_i|| = \frac{7}{5760} \mathrm{c}\, h^5;\ c := \max |f^{(6)}(x_i)|, for\ x_i \in [\mathrm{a,b}]$

Hence, our method applied to second order IVPs is consistent of order five.

4.4. Convergence of the proposed method for the third order (n=3)

Putting n=3 in relations (7)-(9), we get

$$\boldsymbol{P'''(x_{i+c_j}) = f(x_{i+c_j}, P(x_{i+c_j}), P'(x_{i+c_j}), P''(x_{i+c_j}))\ ,\ j = 1,2,} \qquad (18)$$

$$\boldsymbol{P'''_{i+1} = h^3 f(x_{i+1}, P(x_{i+1}),\ P'(x_{i+1}), P''(x_{i+1})),\ i = 0,1,\dots,\mathrm{N}-1} \qquad (19)$$

by using the collocation parameters (9) we can write (18)-(19) as follows:

$$\begin{bmatrix} \frac{12}{5} & -\frac{24}{5} & 3 \\ \frac{65}{2} & -\frac{85}{4} & \frac{149}{24} \\ 60 & -36 & 9 \end{bmatrix} \begin{bmatrix} P_{i+1} \\ hP'_{i+1} \\ h^2P''_{i+1} \end{bmatrix} = \begin{bmatrix} \frac{12}{5} & -\frac{12}{5} & -\frac{3}{5} \\ \frac{65}{2} & \frac{45}{4} & \frac{29}{24} \\ 60 & 24 & 3 \end{bmatrix} \begin{bmatrix} P_i \\ hP'_i \\ h^2P''_i \end{bmatrix} + h^3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{i+c_1} \\ f_{i+c_2} \\ f_{i+1} \end{bmatrix} \quad (20)$$

By solving this system we have

$$\begin{bmatrix} P_{i+1} \\ hP'_{i+1} \\ h^2P''_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_i \\ hP'_i \\ h^2P''_i \end{bmatrix} + h^3 \begin{bmatrix} \frac{215}{56} & -\frac{54}{7} & \frac{97}{24} \\ \frac{200}{21} & -\frac{132}{7} & \frac{59}{6} \\ \frac{25}{2} & -24 & \frac{25}{2} \end{bmatrix} \begin{bmatrix} f_{i+c_1} \\ f_{i+c_2} \\ f_{i+1} \end{bmatrix} \qquad (21)$$

Notice that the matrix of the system (20) nonsingular because

$$\begin{vmatrix} \frac{12}{5} & -\frac{24}{5} & 3 \\ \frac{65}{2} & -\frac{85}{4} & \frac{149}{24} \\ 60 & -36 & 9 \end{vmatrix} = \frac{42}{5} \neq 0$$

and that the all eigenvalues of the matrix:

$$\begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

lie onside to the unit disc.

We obtain from the system (21) that the **local truncation error**:

$$\bar{\tau}_i = \begin{bmatrix} y_{i+1} \\ hy'_{i+1} \\ h^2y''_{i+1} \end{bmatrix} - \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_i \\ hy'_i \\ h^2y''_i \end{bmatrix} - h^3 \begin{bmatrix} \frac{215}{56} & -\frac{54}{7} & \frac{97}{24} \\ \frac{200}{21} & -\frac{132}{7} & \frac{59}{6} \\ \frac{25}{2} & -24 & \frac{25}{2} \end{bmatrix} \begin{bmatrix} y'''(x_{i+c_1}) \\ y'''(x_{i+c_2}) \\ y'''(x_{i+1}) \end{bmatrix} \qquad (22)$$

Using Taylor's expansions of y, y', y'' and substituting in (22), one can estimate the local discretization error as follows:

$$\bar{\tau}_i = h^6 \begin{bmatrix} \frac{137}{14400} \\ \frac{97}{4320} \\ \frac{59}{2160} \end{bmatrix} f^{(6)}(x_i) + O(h^7), i = 1,2, \ldots, \mathrm{N}$$

Note that local discretization error $\bar{\tau}_i$ of this method is from order six, for this using max norm we get global truncation error:

$\bar{\tau}_i = \frac{59}{2160} \mathrm{c}\, h^5;\ c := \max |f^{(6)}(x_i)|, for\ x_i \in [a,b]$

Hence, our method applied to third order IVPs is consistent of order five.

4.5 Absolute Stability of Proposed Method for the first order

To study absolute stability of our method applied to first order IVPs , we will examined as usually by applying them to the Dahlquist's test problem[12]:

$$\boldsymbol{y'} = \lambda\, \boldsymbol{y}, \quad \lambda \in C \qquad (23)$$

Then by using approximations (3) and (4) and collocation parameters $c_1=4/5$, $c_2=11/12$, we have:

$$\boldsymbol{P'(x_{i+c_j}) = \lambda P(x_{i+c_j}), \quad j = 1,2, \quad i = 0,1,\ldots,\mathrm{N}-1},$$

Moreover

$$\boldsymbol{P'_{i+1} = V\, P_{i+1}, \quad P'_i = V\, P_i} \qquad (24)$$

putting $\boldsymbol{V} = (\boldsymbol{h}\lambda)^2$ and using approximations (3) and(5), and the values (24) , we get following formulation matrix:

$\boldsymbol{C(V)}.\underline{\boldsymbol{P}}_{i+1} = \boldsymbol{D(V)}\underline{\boldsymbol{P}}_i$,

where

$$\mathbf{C}(\mathrm{V}) = \begin{bmatrix} \frac{96}{125} - \frac{4544V}{3125} + \frac{512V^2}{3125} & -\frac{8}{125} - \frac{32V}{3125} \\ \frac{605}{3456} - \frac{78287V}{41472} + \frac{6655V^2}{82944} & -\frac{2299}{41472} - \frac{1331V}{497664} \end{bmatrix},$$

$$\mathbf{D}(\mathrm{V}) = \begin{bmatrix} \frac{96}{125} + \frac{1056V}{3125} + \frac{68V^2}{3125} & \frac{4}{125} + \frac{8V}{3125} \\ \frac{605}{3456} + \frac{3025V}{41472} + \frac{55V^2}{27648} & \frac{341}{41472} + \frac{121V}{497664} \end{bmatrix}; \underline{\boldsymbol{P}}_{i+1} \begin{bmatrix} P_{i+1} \\ hP''_{i+1} \end{bmatrix}, \underline{\boldsymbol{P}}_i = \begin{bmatrix} P_i \\ hP''_i \end{bmatrix}$$

Thus, by definition, $\boldsymbol{V}$ belongs to the region of absolute stability of the proposed method if the eigenvalues $\mu_k = \mu_k(V),\ k = 1,2$ of the generalized eigenvalue problem

$$\mu\, \mathbf{C(V)}.\underline{\boldsymbol{X}} = \mathbf{D(V)}.\underline{\boldsymbol{X}}, \quad \underline{\boldsymbol{X}} \neq \mathbf{0},$$

lie inside to the unit disc in the complex plane, i.e. if

$$|\mu_1|, |\mu_2| < \mathbf{1}. \qquad (25)$$

We consider the asymptotic behavior as $\boldsymbol{V} \to \infty$, multiplying the characterizing equation by $\boldsymbol{V}^{-3}$, we get

$$\lim_{V\to\infty} V^{-3} \det(\mu C(V) - D(V)) = \begin{vmatrix} -\frac{68}{3125} + \frac{512\mu}{3125} & -\frac{8}{3125} - \frac{32\mu}{3125} \\ -\frac{55}{27648} + \frac{6655\mu}{82944} & -\frac{121}{497664} - \frac{1331\mu}{497664} \end{vmatrix}$$

$$\Pi(\mu) = \frac{77}{388800000} + \frac{79079\mu}{388800000} + \frac{9317\mu^2}{24300000}, \qquad (26)$$

it is characterizing equation, two roots $\mu_1 = -0.5295, \mu_2 = -0.00098$. This fulfills the condition of the roots (25) and whomever we conclude is that our method applied to first order IVPs is absolutely stable.

4.6 Absolute Stability of second the QSC Methods

In the same manner for studying absolute stability, we apply our method to the test equation:

$$\boldsymbol{y}'' = \lambda^2\, \boldsymbol{y}, \quad \lambda \in C \qquad (27)$$

Then by using approximations (3) and (5) and collocation parameters c_1=4/5, c_2=11/12, we have:

$$\boldsymbol{P''(x_{i+c_j}) = \lambda^2 P(x_{i+c_j}),\quad j = 1,2,\quad i = 0,1,...,N-1},$$

Moreover

$$\boldsymbol{P''_{i+1} = V\, P_{i+1},\quad P''_i = V\, P_i} \tag{28}$$

putting $\boldsymbol{V = (h\lambda)^2}$ and using approximations (3) and(5), and the values (28) , we get following formulation matrix:

$$\boldsymbol{C(V).\underline{P}_{i+1} = D(V)\underline{P}_i},$$

where

$$\mathbf{C}(\mathrm{V}) = \begin{bmatrix} -\frac{144}{25} + \frac{3444V}{3125} - \frac{32V^2}{3125} & \frac{96}{25} - \frac{512V}{3125} \\ -\frac{275}{72} + \frac{25949V}{41472} - \frac{1331V^2}{497664} & \frac{341}{144} - \frac{6655V}{82944} \end{bmatrix},$$

$$\mathbf{D}(\mathrm{V}) = \begin{bmatrix} -\frac{144}{25} + \frac{444V}{3125} + \frac{8V^2}{3125} & -\frac{48}{25} - \frac{68V}{3125} \\ -\frac{275}{72} + \frac{6941V}{41472} + \frac{121V^2}{497664} & -\frac{209}{144} - \frac{55V}{27648} \end{bmatrix}; \underline{\boldsymbol{P}}_{i+1} \begin{bmatrix} P_{i+1} \\ hP'_{i+1} \end{bmatrix}, \underline{\boldsymbol{P}}_i = \begin{bmatrix} P_i \\ hP'_i \end{bmatrix}$$

And so on, $\boldsymbol{V}$ belongs to the region of absolute stability of the proposed method if the eigenvalues $\mu_k = \mu_k(V),\ k = 1,2$ of the generalized eigenvalue problem

$$\mu\, C(Z).\underline{X} = D(Z).\underline{X},\quad \underline{X} \neq 0,$$

lie inside to the unit disc in the complex plane, i.e. if

$$|\mu_1|, |\mu_2| < \mathbf{1}. \tag{29}$$

we consider the asymptotic behavior as $\boldsymbol{V} \to \infty$, multiplying the characterizing equation by $\boldsymbol{V}^{-3}$, we get

$$\lim_{V\to\infty} V^{-3}\det(\mu C(V) - D(V)) = \begin{vmatrix} -\frac{8}{3125} - \frac{32\mu}{3125} & \frac{68}{3125} - \frac{512\mu}{3125} \\ -\frac{121}{497664} - \frac{1331\mu}{497664} & \frac{55}{27648} - \frac{6655\mu}{82944} \end{vmatrix} \Rightarrow$$

$$\Pi(\mu) = \frac{77}{388800000} + \frac{79079\mu}{388800000} + \frac{9317\mu^2}{24300000},$$

It is the same characterizing equation, which we obtained in the previous relation (26) , where two their roots $\mu_1 = -0.5295, \mu_2 = -0.00098$. Then the condition of roots is fulfilled and hence our method applied to secod order IVPs is absolutely stable.

4. 7 Absolute Stability of Third the QSC Methods

In the same manner for studying absolute stability, we apply our method to the test equation:

$$\boldsymbol{y''' = \lambda^3\, y},\quad \lambda \in C \tag{30}$$

Then by using approximations (3) and (5) and collocation parameters c_1=4/5, c_2=11/12, c_3=1 we have:

$$P'''(x_{i+c_j}) = \lambda^3 P(x_{i+c_j}), \quad j = 1,2,3 \quad i = 0,1,\ldots,N-1,$$

putting $V = (h\lambda)^3$ and using approximations (3) and(6) , we get following formulation matrix:

$$C(V).\underline{P}_{i+1} = D(V)\underline{P}_i,$$

where

$$\mathbf{C}(\mathrm{V}) = \begin{bmatrix} \frac{12}{5} - \frac{2944V}{3125} & -\frac{24}{5} + \frac{512V}{3125} & 3 - \frac{32V}{3125} \\ \frac{65}{2} - \frac{41261V}{41472} & -\frac{85}{4} + \frac{70543V}{414720} & \frac{149}{24} - \frac{1331V}{497664} \\ 60 - V & -36 & 9 \end{bmatrix},$$

$$\mathbf{D}(\mathrm{V}) = \begin{bmatrix} \frac{12}{5} - \frac{181V}{3125} & -\frac{12}{5} - \frac{68V}{3125} & -\frac{3}{5} - \frac{8V}{3125} \\ \frac{65}{2} - \frac{211V}{41472} & \frac{45}{4} - \frac{55V}{27648} & \frac{29}{24} - \frac{121V}{497664} \\ 60 & 24 & 3 \end{bmatrix}; \; \underline{P}_{i+1} \begin{bmatrix} P_{i+1} \\ hP'_{i+1} \\ hP''_{i+1} \end{bmatrix}, \underline{P}_i = \begin{bmatrix} P_i \\ hP'_i \\ hP''_i \end{bmatrix}$$

And so on, V belongs to the region of absolute stability of the proposed method if the eigenvalues $\mu_k = \mu_k(V)$, $k = 1,2,3$ of the generalized eigenvalue problem

$$\mu C(Z).\underline{X} = D(Z).\underline{X}, \quad \underline{X} \neq 0,$$

lie inside to the unit disc in the complex plane, i.e. if

$$|\mu_1|, |\mu_2|, |\mu_3| < 1. \qquad (31)$$

we consider the asymptotic behavior as $V \to \infty$, multiplying the characterizing equation by V^{-3}, we get

$$\lim_{V\to\infty} V^{-3} \det(\mu C(V) - D(V)) = \begin{vmatrix} \frac{181}{3125} - \frac{2944\mu}{3125} & \frac{68}{3125} + \frac{512\mu}{3125} & \frac{8}{3125} - \frac{32\mu}{3125} \\ \frac{211}{41472} - \frac{41261\mu}{41472} & \frac{55}{27648} + \frac{70543\mu}{414720} & \frac{121}{497664} - \frac{1331\mu}{497664} \\ -\mu & 0 & 0 \end{vmatrix}$$

$$\Pi(\mu) = -\frac{77\mu}{388800000} + \frac{842611\mu^2}{1944000000} - \frac{158389\mu^3}{121500000},$$

It is the characterizing equation, where three their roots $\mu_1 = 0.33203$, $\mu_2 = 0.000457, \mu_3 = 0$.

Then the condition of roots is fulfilled and hence our method is absolutely stable when is applied to differential equations of the third order.

5. Numerical Results

In this section, we solve several linear and non-linear problems to test the proposed method. All comparisons of the results, our method, with the results of the other methods were computed with the same conditions and criteria. The numerical experiments were performed in double precision using Mathematica.

Problem 1 [**3**]: Firstly, we consider the first linear problem:

$$y'(x) = \frac{1}{2}(1-y), \qquad y(0) = \frac{1}{2}$$

Theoretical solution is given

$$y(x) = 1 - \frac{e^{-x/2}}{2}$$

We solve the problem by the proposed method, we compare our results with the results of hybrid block method in [**3**] and put the results in Table 1. In Figure 1, we draw the solution by our method with the exact solution.

Table 1: Comparison of absolute errors in the solutions of problem1.

x	Hybrid Block Method [3]		Our Method.	
	Abs. Error	The solution *y*	The solution *y*	Abs. Error
0.1	1.218026E-13	0.524385287750861	0.5243852877496415	1.44328991 E-15
0.2	1.399991E-13	0.547581290981880	0.5475812909820175	2.77555755 E -15
0.3	1.184941E-12	0.569646011786286	0.569646011787467	4.10782519 E -15
0.4	1.538991E-12	0.590634623462548	0.5906346234610039	5.16253706 E -15
0.5	1.110001E-12	0.610599608463187	0.6105996084642914	6.16173773 E -15
0.6	5.270229E-12	0.629590889658614	0.629590889659134	7.04991620 E -15
0.7	2.10898E-12	0.647655955142752	0.6476559551406353	7.93809462 E -15
0.8	1.297895E-11	0.664839976969201	0.6648399769821717	8.65973959 E -15
0.9	3.08229E-11	0.681185924158290	0.681185924189104	9.38138455 E -15
1.0	4.121925E-11	0.696734670139561	0.6967346701436734	9.88098491 E -15
2.0	-----	------	0.8160602794142667	1.21014135 E-14
3.0	-----	-------	0.888434919925774	1.11438636 E-14

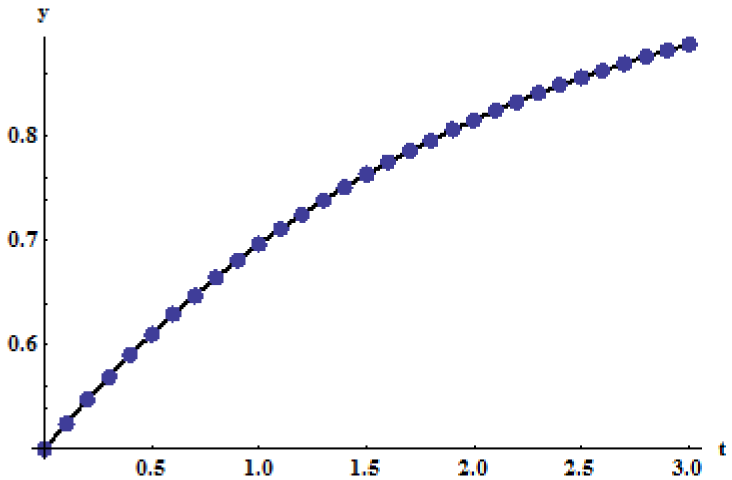

Figure 1: Both the solution by our method with the exact solution.

Problem 2 [**12**]: We consider an autonomous on linear problem for which there is a singular feature due to the presence of a pole on the derivatives of the solution:

$$y'(x) = \frac{1}{y^2(x)}, \qquad y(-1) = \sqrt[-3]{3}$$

Theoretical solution is given

$$y(x) = \frac{(10 + 9x)^{1/3}}{3^{1/3}}$$

We solve the problem by the proposed method, we compare our results with the results of Ramos et al. method in [**12**] and put the results in Table 2.

Table 2: Results of Problem 2 solved with the proposed method and other method.

n	Ramos et al. [12, 2015]	Our Method
	Max. Error y	Max. Error y
182	8.82844 E-06	6.80504 E-10
289	2.02776 E-06	1.64148 E-11
462	4.36543 E-07	3.03781 E-11
635	7.71190 E-08	5.04281 E-11
975	1.92111 E-08	7.96219 E-11
1437	4.22068 E-09	1.23991 E-12
2119	1.03579 E-09	2.95244 E-12
3113	3.77049 E-10	9.22623 E-13

Problem 3 [**4**]: We examine now the linear second-order differential equation:

$$y''(x) - y(x) = x - 1, \qquad y(0) = 2, \qquad y'(0) = -2$$

Theoretical solution is given

$$y(x) = 1 - x + e^{-x}$$

We solve the problem3 using by our proposed method, and the absolute errors are compared with those produced by Bilesanmi et al. [**4**, 2019] as shown in Table 3. The comparison shows that our approach gave better result compared with other method.

Table 3: Results of Problem 3 solved with the proposed method and other method.

x	Bilesanmi et al. [4, 2019]	Our Method
	Abs. Error y	Abs. Error y
0.1	2.591705 E−12	1.587618925213973 E−14
0.2	5.964562E−12	4.762856775641921 E−14
0.3	9.366508E−12	9.22595333463505 E−14
0.4	1.286815E−11	1.495470414170085 E−13
0.5	1.649259E−11	2.164934898019055 E−13
0.6	2.029099E−11	2.948752353404416 E−13
0.7	2.428346E−11	3.857469899060106 E−13
0.8	2.852540E−11	4.879985304739875 E−13
0.9	3.304634E−11	6.014078124394473 E−13
1.0	3.792694E−11	7.253087019876148 E−13
2.0	-----	2.626565631658195 E−12
3.0	------	6.516238937326335 E−12

Problem 4: We consider the nonlinear second order problem [**11**]

$$\boldsymbol{y''(x) = y^2(x) + \frac{1}{2}\cos x - \sin^4(\frac{x}{2}),}$$

$y(0) = y'(0) = 0$.

The theoretical solution is.

$y(x) = sin^2(\frac{x}{2})$.

The maximum absolute errors in $y(x)$ and $y'(x)$, the right side is our results and the left side is a computational algorithm [11] are given in Table 4. We draw both the solution by our method with the exact solution in Figure 2. Our numerical results and other results are summarized in Table 4.

Table 4: Results of Problem 4 solved with the proposed method and other method.

n	A computational algorithm [11, 2018]		Our Method	
	Max. Error y'	Max. Error y	Max. Error y	Max. Error y'
128	3.083482 E-02	2.283364 E-03	4.120907 E-09	2.567807 E-08
256	2.080867 E-02	1.416236 E-03	8.534215 E-10	5.076537 E-09
512	1.396590 E-03	8.000433 E-04	1.243802 E-11	3.983322 E-10
1024	9.322166E-03	4.311204 E-04	9.675840 E-12	8.193601 E-11
2048	6.191766 E-03	2.279877 E-04	2.453619 E-12	1.907956 E-11

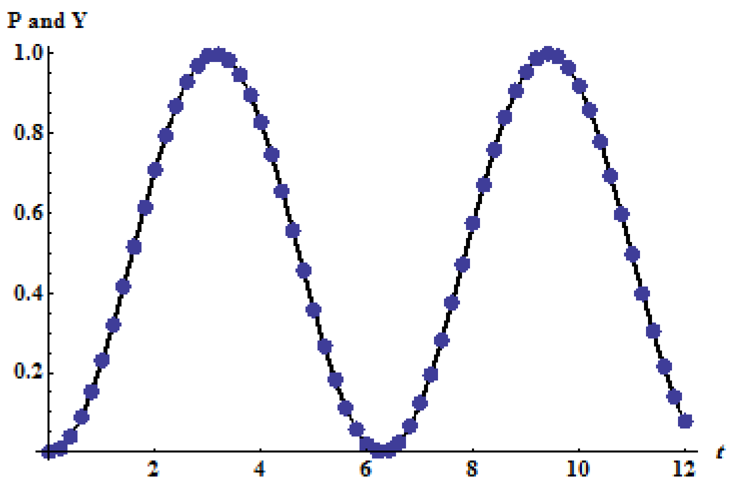

Figure 2: Both the solution and the exact solution by our method.

Problem 5: we examine a general the third-order linear problem [**5**]

$$y'''(x) + 2y''(x) - 9y'(x) - 18y(x) = -18x - 18x^2 + 22,$$

$y(0) = -2, y'(0) = -8, y''(0) = -12$.

The theoretical solution is

$y(x) = -2e^{3x} + e^{-2x} + x^2 - 1$.

The maximum absolute errors in $y(x)$ are computed, where the right side is our results and the left side is a hybrid numerical method [**5**] are given in Table 5.

h	Hybrid Numerical Method [5, 2019]		Our Method	
	y	Max error	y	Max error
1/10	−40.035738498925235739031 6	6.421349E(−8)	-40.0357385675764326765432	4.437708E-9
1/20	−40.035738562139319970150 3	9.994028E(−10)	-40.0357385627164655879321	4.222542E-10
1/40	−40.035738563123122652492 9	1.560014E(−11)	-40.03573856312512564321532	1.35921E-11
1/80	−40.035738563138479180447 8	2.436095E(−13)	-40.03573856313854329087654	1.77636E-13
1/160	−40.035738563138719182996 7	3.606966E(−15)	-40.03573856313872432098	1.5310E-15

6. Conclusions and Recommendations

We have presented the method that is efficient, stable and convergent for solving first, second and third order initial value problems in ordinary differential equations. This proposed method is tested by solving five problems, the results were effective and accurate, and comparisons of our proposed method with five other methods indicate the preference of our results.

We recommend that this method should be used to solve ordinary differential equations from first, second and third order. Furthermore, we suggest developing new methods for solving problems in ordinary differential equations of high order.

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