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Large Deflection Analysis of Thin Plates Using Modified Finite Element Method

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□ ABSTRACT

In this study, thin plates are analyzed using modified finite element displacement approach in the geometrical nonlinear case (large deflections , small rotations). The displacement approach is based on the incremental form of the principle of virtual displacement. Wherein the displacement function of the rectangular plate element (ACM) which has 12 degrees of freedom, is modified by adding new terms. The new terms represents the effect of the external load. As a result, shape functions are classified into two categories the first one is the homogenous shape functions relevant to degrees of freedom and the second is the nonhomogeneous shape functions related to the element loading. Stiffness matrices, external force vector and new terms from applying the modified finite element is concluded.

Some examples are computed and plotted and the results of the displacements, stresses, strains and internal forces are compared with those produced by MSC Patran program as well as the analytical solution. The comparison showed that the modified method presented an improved solution.

Keywords : Finite element method, Geometric nonlinear analysis, Geometric stiffness matrix, Large deflection analysis of thin plates, Modified approach.

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تحليل البلاطات الرقيقة ضمن حالة السهوم الكبيرة باستخدام طريقة العناصر المنتهية المعدلة

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□ ملخّص□

تم في هذه الدراسة نمذجة البلاطات الرقيقة و تحليلها عبر اعتماد مقاربة معدلة لطريقة العناصر المنتهية – نموذج الانتقالات – و ذلك ضمن المجال اللاخطي الهندسي (انتقالات كبيرة ، دورانات صغيرة). تم اعتماد المنهج التزايدي التكراري في الحل و تطبيقه على مبدأ الانتقالات الوهمية. إذ تم تعديل تابع الانتقال في طريقة الانتقالات التقليدية لعنصر البلاطة المستطيل ذي 12 درجة حرية المعروف بـ (ACM) و إضافة حدود جديدة تأخذ تأثير الحمولة الخارجية المؤثرة على البلاطة بعين الاعتبار، و نتيجة لذلك أصبحت توابع الشكل مؤلفة من جزأين : جزء متجانس متعلق بدرجات الحرية للعنصر و جزء آخر غير متجانس مرتبط بالحمولات الخارجية المؤثرة على العنصر. و تم استنتاج مصفوفات الصلابة، شعاع القوى الخارجية و الحدود الإضافية الناتجة عن الطريقة المعدلة.

تم دراسة بعض الأمثلة و رسم مخططات السهوم و القوى الداخلية و تمت مقارنة هذه النتائج مع القيم الناتجة عن التحليل باستخدام برنامج MSC Patran و الحل التحليلي. بينت النتائج أن الكود البرمجي المطور للطريقة المعدلة أعطى قيم أقرب للحل الدقيق من برنامج MSC Patran و ذلك على الرغم من استخدام شبكة خشنة عند تقسيم البلاطة (عدد عناصر صغير نسبياً).

الكلمات المفتاحية : طريقة العناصر المنتهية، تحليل لا خطي هندسي، مصفوفة الصلابة الهندسية، تحليل البلاطات الرقيقة حالة السهوم الكبيرة، مقاربة معدلة.

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Background

Thin plates are relatively sensitive and deformable members. They highly change their geometric shape under the influence of external lateral loads because of the small thickness, so it is well recommended to analyze thin plates taking into account geometric nonlinear effects and avoid as far as possible linear analysis. In addition to the above, nonlinear analysis reduces the weight of thin plates and ensures optimal structural design [1]. One of the most common methods to carry out geometric nonlinear analysis is the finite element method.

According to Addisu [2], the strong development of the method from engineer's point of view has been led by Zienkiwicz and Taylor [3]. The mathematical theory of the finite elements has been developed and promoted by many scientists. Among them one can mention Strang and Fix [4], Babuska and Aziz [5], Oden and Reddy [6].

Argyris, Clough and Martin were among the first pioneers who worked in FEM and they have played a major role in developing this method and achieving its wide popularity [7]. In 1965, Argyris [8] derived geometric stiffness matrix in natural coordinate system but he didn't provide any examples or engineering applications. Martin [9] used the principle of virtual displacement taking into consideration the nonlinear terms in deformation-displacement relations and derived the initial stress matrix for triangular finite element then he used this matrix with linear stiffness matrix through an incremental iterative procedure to solve large deflection problems in thin plates. Murray and Wilson [10] used triangular plate element also through incremental iterative procedure and their study was applicable to shells with large deflections as well, results obtained were close to analytical solutions of thin plates. Melliere [11] developed an eighteen degrees-of-freedom triangular element to derive geometric stiffness matrix, then conjunct it with the standard small displacement 'elastic' stiffness matrix in linear-incremental approach to obtain numerical solutions for the large displacement problem of thin elastic plates and shells.

Recently, Radek Gabbasov et al [12] investigated the large deflection analysis of thin plates by proposing a numerical method basing on the use of difference equations of successive approximation method (MSA).

In the present work, large deflection analysis of thin plates is investigated by applying a modified version of the well-known ACM rectangular plate element, see for example [13, 14, 15, 16], using a modified FEM – displacement approach. This approach was presented in many international events. The oldest events were the first international workshop on Trefftz method-recent development and perspectives hold in Cracow, Poland (1996) and the second international workshop on Trefftz Method high Performance, Global, Sub-Domain and Finite Element Formulations hold in Sintra, Portugal (1999). The modified finite element displacement approach was described in details, among other approaches, in report [17], in many conference papers [18,19,20,21,22,23], and in journal publications [24].

We aren't going to introduce a long historical background about thin plates in nonlinear domain but for readers who looking for more recent references you can see [7,12,27,28,29] which represents a sample of important papers that contributed significantly in the development of nonlinear analysis of thin plates in last ten years.

The finite element displacement approach presented here, use displacement approximation basis separated into homogeneous part and particular part. The parametric form chosen fulfils strictly the differential equation of the Kirchhoff's plate. The homogeneous part satisfies the homogeneous part of the differential equation and the added particular part satisfies the non-homogeneous differential equation. The added particular terms enables considering the effect of the external loading at the finite element level. The free parameters are related to the degrees of freedoms of the element in an analogous way adopted in the conventional finite element displacement approach. As a result, the shape functions are separated into two parts, a homogenous part related to the degrees of freedom of the element and a new non-homogenous part dependent on the element loading at the finite element level. For the convenience of the reader the way of constructing the displacement basis will here be recalled. Stiffness matrix, external equivalent loading vector and new terms resulting from the application of the modified finite element method will be concluded in details.

The application of the modified finite element approach is based on the incremental form of the principle of virtual displacement. A simple iterative incremental procedure is adopted in order to analysis some plate examples of different boundary conditions. The results are compared with those produced by MSC Patran [25] as well as an analytical solution [26], and showed that the modified method presented an improved solution.

Basic Formulations

A rectangular plate element with 12 DOFs is shown in Fig.1, Based on Kirchhoff-Love theory for thin plates, the displacement components u, v and w in the x,y,z directions in a plate element can be expressed in terms of the corresponding mid-plane displacement components u^0, v^0, w^0 and the rotations θ_x, θ_y of the mid-plane normal along x and y axis :

$$u(x,y,z) = u^0(x,y) + z.\theta_x(x,y) \qquad (1)$$

$$v(x,y,z) = v^0(x,y) + z.\theta_y(x,y) \qquad (2)$$

$$w(x,y,z) = w^0(x,y) \qquad (3)$$

Where :

$$\theta_x = \frac{\partial w}{\partial y} \quad ; \quad \theta_y = -\frac{\partial w}{\partial x} \qquad (4)$$

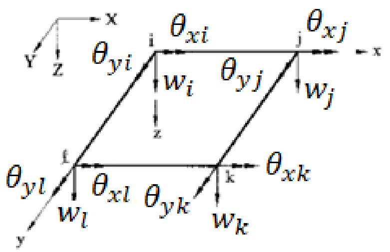


Fig. 1 : Degrees Of Freedom For Thin Plate [1]

For large deflection analysis, the strain vector at any point of the rectangular plate element is

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}.\left(\frac{\partial w}{\partial x}\right)^2 \qquad (5)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}.\left(\frac{\partial w}{\partial y}\right)^2 \qquad (6)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x}\right).\left(\frac{\partial w}{\partial y}\right)\right) \qquad (7)$$

$$\varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = 0 \qquad (8)$$

Where : $\{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}\}$: are the normal and shear strains respectively.

Substituting Eqs. (1,2,3,4) into (5,6,7) the strain can be written as

$$\varepsilon_{ij} = \eta_{ij} + z.\chi_{ij} \qquad (9)$$

In which

$$\varepsilon_{ij} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix};\ \chi_{ij} = \begin{Bmatrix} \chi_{xx} \\ 2.\chi_{xy} \\ \chi_{yy} \end{Bmatrix};\ \eta_{ij} = \begin{Bmatrix} \eta_{xx} \\ 2.\eta_{xy} \\ \eta_{yy} \end{Bmatrix} \quad (10)$$

$$\eta_{ij} = \begin{Bmatrix} \frac{1}{2}.\left(w_{,x}\right)^2 \\ \frac{1}{2}.\left(w_{,y}\right)^2 \\ w_{,x}.w_{,y} \end{Bmatrix};\ \chi_{ij} = -\begin{Bmatrix} w_{,xx} \\ w_{,yy} \\ w_{,x}.w_{,y} \end{Bmatrix} \quad (11)$$

For thin plates, the constitutive relationship can be expressed as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yy} \end{Bmatrix} = \frac{E}{1-\vartheta^2}.\begin{bmatrix} 1 & 0 & \vartheta \\ 0 & \frac{1-\vartheta}{2} & 0 \\ \vartheta & 0 & 1 \end{bmatrix}.\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{xy} \\ \varepsilon_{yy} \end{Bmatrix} \quad (12)$$

Where :

$\{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}$: are the normal and shear stresses respectively.

Internal forces for differential element in thin plate is shown in Fig. 2 and the internal forces for thin plates are given as following

$$\begin{Bmatrix} M_x \\ M_{xy} \\ M_y \end{Bmatrix} = \frac{E.h^3}{12(1-\vartheta^2)}.\begin{bmatrix} 1 & 0 & \vartheta \\ 0 & \frac{1-\vartheta}{2} & 0 \\ \vartheta & 0 & 1 \end{bmatrix}.\begin{Bmatrix} \chi_{xx} \\ 2.\chi_{xy} \\ \chi_{yy} \end{Bmatrix} \quad (13)$$

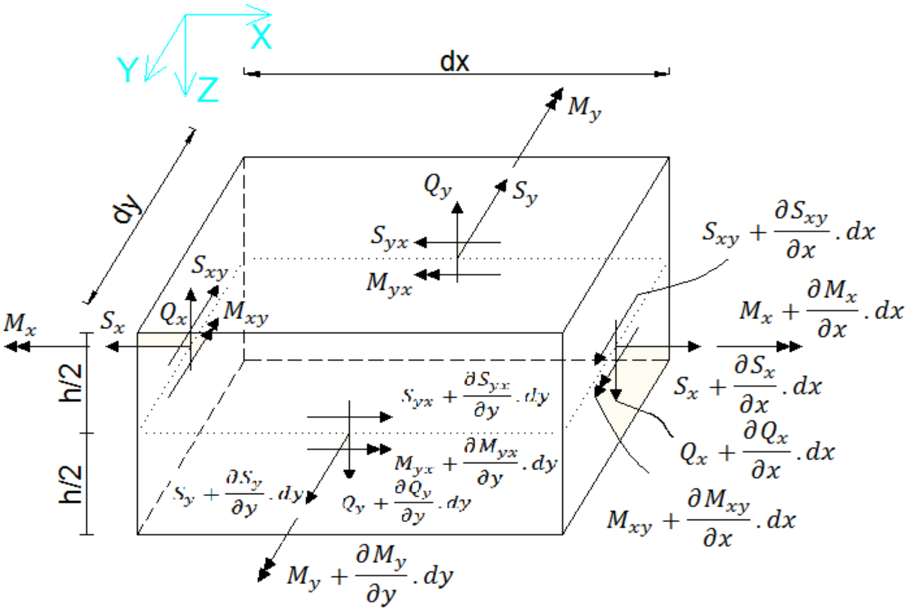


Fig. 2 : Internal Forces For Differential Element In Thin Plat [1]

$$\begin{Bmatrix} S_x \\ S_{xy} \\ S_y \end{Bmatrix} = \frac{E.h}{1-\vartheta^2}.\begin{bmatrix} 1 & 0 & \vartheta \\ 0 & (1-\vartheta)/2 & 0 \\ \vartheta & 0 & 1 \end{bmatrix}.\begin{Bmatrix} \frac{1}{2}.\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2}.\frac{\partial w}{\partial x}.\frac{\partial w}{\partial y} \\ \frac{1}{2}.\left(\frac{\partial w}{\partial y}\right)^2 \end{Bmatrix} \quad (14)$$

Where :

$\{S_x, S_{xy}, S_y\}$: are the membrane forces vectors in the mid-plane, $\{M_x, M_{xy}, M_y\}$ are the bending moment vectors.

$D = \frac{E.h^3}{12(1-\vartheta^2)}$: is the bending stiffness coefficient for thin plate.

The governing differential equations for large deflections of thin plates according to Von Karman formulation :

$$D.\nabla^4 w = q(x,y) + h.\left(-2.\frac{\partial^2 \emptyset}{\partial x.\partial y}.\frac{\partial^2 w}{\partial x.\partial y} + \frac{\partial^2 \emptyset}{\partial y^2}.\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial x^2}.\frac{\partial^2 w}{\partial y^2}\right) \quad (15)$$

$$\nabla^4 \emptyset = E.\left(\left(\frac{\partial^2 w}{\partial x.\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2}\right) \quad (16)$$

Where :

$\emptyset$: is the stress function.

$\nabla^4(\,) = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial x^2.\partial y^2} + \frac{\partial^4}{\partial y^4}$: is called the biharmonic operator.

These two Eqs (15,16) are coupled, nonlinear, partial differential equations, each of fourth order. Unfortunately, there isn't any exact solution for these Eqs, also the analytical solutions based on Navier's and Levy's methods can't be used to derive stiffness matrices in finite element approach, so we are unable to use these two Eqs in the modified method presented in this study. We formulated an approximation function for the deflection that satisfy the differential equation of the linear analysis of the Kirchhoff's plate, which is

$$D.\nabla^4 w = q(x,y) \qquad (17)$$

And used it in the non linear case. The following parametric form of the deflection satisfies the homogenous part of the differential equation.

$$w(x,y) = c_0 + c_1.x + c_2.y + c_3.x^2 + c_4.x.y + c_5.y^2 + c_6.x^3 + c_7.x^2.y + c_8.x.y^2 + c_9.y^3 + c_{10}.x^3.y + c_{11}.x.y^3 \qquad (18)$$

In the framework of the modified finite element method, the parametric form is extended in order to consider the element loading at the finite element level.

Formulation Of The Modified Method With The Particular Terms

1- Modification Of Displacement Function And Shape Functions

In order to capture the effect of the external loading at the finite element level the parametric form is extended to contain fifteen parameters instead of twelve.

$$w(x,y) = c_0 + c_1.x + c_2.y + c_3.x^2 + c_4.x.y + c_5.y^2 + c_6.x^3 + c_7.x^2.y + c_8.x.y^2 + c_9.y^3 + c_{10}.x^3.y + c_{11}.x.y^3 + c_{12}.x^2.y^2 + c_{13}.x^4.y + c_{14}.x.y^4 + c_{15}.x^3.y^3 \qquad (19)$$

In turns the element loading with different intensities at the element nodes is approximated by the following parametric form :

$$q(x,y) = \{1 \quad x \quad y \quad x.y\}.\begin{Bmatrix} c_{\bar{q}^1} \\ c_{\bar{q}^2} \\ c_{\bar{q}^3} \\ c_{\bar{q}^4} \end{Bmatrix} \qquad (20)$$

By substituting displacement function in the differential equation (17), then taking the load function as in Eq (20) and matching both sides of the resulting relations some of the parameters can be related to the load intensity.

$$\begin{Bmatrix} c_{12} \\ c_{13} \\ c_{14} \\ c_{15} \end{Bmatrix} = \frac{1}{D}.\begin{Bmatrix} \frac{c_{\bar{q}^1}}{8} \\ \frac{c_{\bar{q}^2}}{24} \\ \frac{c_{\bar{q}^3}}{24} \\ \frac{c_{\bar{q}^4}}{72} \end{Bmatrix} \qquad (21)$$

By substituting the nodal coordinates in the load function and taking the inverse then substituting in Eq (19), we obtain the modified displacement function

$$w_i = M_i^k.c_k + \bar{M}_{ij}.\bar{p}^j \qquad (22)$$

$$M_i^k = \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ x.y \\ y^2 \\ x^3 \\ x^2.y \\ x.y^2 \\ y^3 \\ x^3.y \\ x.y^3 \end{bmatrix}^T \; ; \; c_k = \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \end{Bmatrix} \; ; \; \bar{p}^j = \begin{Bmatrix} \bar{q}^{(1)} \\ \bar{q}^{(2)} \\ \bar{q}^{(3)} \\ \bar{q}^{(4)} \end{Bmatrix} \quad (23)$$

$$\bar{M}_{ij} = \frac{1}{D}.\left\{\frac{x^2.y^2}{8} \quad \frac{x^4.y}{24} \quad \frac{x.y^4}{24} \quad \frac{x^3.y^3}{72}\right\}.[\bar{A}] \quad (24.a)$$

$$[\bar{A}] = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} & \bar{A}_{14} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} & \bar{A}_{24} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} & \bar{A}_{34} \\ \bar{A}_{41} & \bar{A}_{42} & \bar{A}_{43} & \bar{A}_{44} \end{bmatrix} \quad (24.b)$$

Where

$[\bar{A}]$: is the matrix depend on the element geometry and the material properties.

In the special case, when load function is constant (for example distributed load) $q(x,y) = \bar{q}$, the modified displacement function in its two parts (homogeneous and non-homogeneous) and in the absence of external moments $(m_x \; , \; m_y)$ becomes as follows :

$$w(x,y) = \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ x.y \\ y^2 \\ x^3 \\ x^2.y \\ x.y^2 \\ y^3 \\ x^3.y \\ x.y^3 \end{bmatrix}^T . \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \end{Bmatrix} + \frac{x^2.y^2}{8.D}.\bar{q} \quad (25)$$

In order to find shape functions, we apply the algorithm of FEM on two parts of displacement function. The homogeneous part is well known in literatures [1,10], while the non-homogeneous part is as follows

$$w_i = N_i^{m(e)}.u_{m(e)} + \bar{N}_{ij}.\bar{p}^j \quad (26)$$

$N_i^{m(e)}$: is the homogeneous part of shape functions.

$\bar{N}_{ij}$: is the non-homogeneous part of shape functions which is given in detail in Appendix A.

$u_{m(e)}$: are the nodal displacements.

$\bar{p}^j$: are the nodal forces.

2- Incremental Form Of The Principle Of Virtual Displacement

Using the incremental form of the principle of the virtual displacement as a variational basis for FEM-displacement approach requires that the first total energy variation of the deformed domain in an increment is equivalent to zero

$$\delta\Delta\pi = 0 \qquad (27)$$

From a given state of equilibrium the corresponding u_i^0 displacements are replaced by a new displacement state u_i with a small difference

$$\tilde{u}_i = u_i^0 + u_i \rightarrow \tilde{u}_{i+1} = \tilde{u}_i + \Delta u_{i+1} \qquad (28)$$

u_i : increment in displacement.

$\tilde{u}_i$: unknown situation for displacement.

this lead to a small increment in stresses and strains

$$\varepsilon_{ij} = \varepsilon_{ij}^0 + \Delta\varepsilon_{ij} \; ; \;\; \sigma_{ij} = \sigma_{ij}^0 + \Delta\sigma_{ij} \qquad (29)$$

ε_{ij}^0 , σ_{ij}^0 : strain and stress tensors for initial state (linear) respectively.

$\Delta\varepsilon_{ij}$, $\Delta\sigma_{ij}$: small increments in strain and stress tensors.

The principle of virtual displacement becomes

$$\int_V \delta(\Delta\varepsilon_{ij}) c^{ijkl} \varepsilon_{kl}^0 dV + \int_V \delta(\Delta\varepsilon_{ij}) c^{ijkl} \Delta\varepsilon_{kl} dV = \int_V \tilde{f}_i \delta(\Delta w_i) dV \qquad (30)$$

$$\sigma_{ij} = c^{ijkl} . \varepsilon_{ij} \qquad (31)$$

c^{ijkl} : is the elasticity tensor.

Transforming from the 3D-form to the 2D thin plate form, the Eq(31) becomes

$$\int_A \delta(\Delta\varepsilon_{ij}) c^{ijkl} \varepsilon_{kl}^0 dA + \int_A \delta(\Delta\varepsilon_{ij}) c^{ijkl} \Delta\varepsilon_{kl} dA = \int_A \tilde{q}_i \delta(\Delta w_i) dA \qquad (32)$$

The left side of the Eq(32) consists of four main terms they are :

$$T_1 = \int_A \delta(\Delta\varepsilon_{ij}) c^{ijkl} \varepsilon_{kl}^0 dA$$

$$= \int_A \delta\Delta(N_{,ij}.\bar{w} + \bar{N}_{,ij}.\bar{p}^j). c^{ijkl}. \Delta(N_{,kl}.\bar{w} + \bar{N}_{,kl}.\bar{p}^l) dA \qquad (33)$$

Where

$$\varepsilon_{ij} = N_{,ij}.\bar{w} + \bar{N}_{,ij}.\bar{p}^j \; ; \; \varepsilon_{kl} = N_{,kl}.\bar{w} + \bar{N}_{,kl}.\bar{p}^l \qquad (34)$$

$N_{,ij}$, $N_{,kl}$: are the second derivative of the shape functions (homogeneous part).

$\bar{N}_{,ij}$, $\bar{N}_{,kl}$: are the second derivative of the shape functions (non-homogeneous part).

By multiplying the brackets and noticing that the terms $\left(\left\{\int_A \delta(\bar{N}_{,ij}.\bar{p}^j). C_{ijkl}. N_{,kl}. dA\right\} = 0\right)$ and $\left(\left\{\int_A \delta(\bar{N}_{,ij}.\bar{p}^j). C_{ijkl}. (\bar{N}_{,kl}.\bar{p}^l). dA\right\} = 0\right)$ because when we took first variable of nodal displacement they were known values, and we know that variable of known value is zero.

$$T_1 = \left[\delta\bar{w}.\left\{\int_A N_{,ij}. c^{ijkl}. N_{,kl}. dA\right\}.\bar{w} + \delta\bar{w}.\left\{\int_A N_{,ij}. c^{ijkl}. (\bar{N}_{,kl}.\bar{p}^l). dA\right\}\right] \qquad (35)$$

Where

$K_E = \left\{\int_A N_{,ij}. c^{ijkl}. N_{,kl}. dA\right\}$: represents the linear bending stiffness matrix for thin plate.

$R_1 = \left\{\int_A N_{,ij}. c^{ijkl}. (\bar{N}_{,kl}.\bar{p}^l). dA\right\}$: new term using the modified FEM.

$$T_2 = T_3 = \frac{1}{2}.\int_A \delta\Delta\left(N_{,i}.\bar{w} + \bar{N}_{,i}.\bar{p}^i\right).M^{ij}.\Delta\left(N_{,j}.\bar{w} + \bar{N}_{,j}.\bar{P}^j\right)dA \qquad (36)$$

$N_{,i}$, $N_{,j}$: are the first derivatives of the shape functions (homogeneous part).

$\bar{N}_{,i}$, $\bar{N}_{,j}$: are the first derivative of the shape functions (non-homogeneous part).

Also by multiplying terms between brackets and applying the same above notice

$$T_2 = \frac{1}{2}.\left[\delta\bar{w}.\left\{\int_A N_{,i}.M^{ij}.N_{,j}.dA\right\}.\bar{w} + \delta\bar{w}.\left\{\int_A N_{,i}.M^{ij}.\left(\bar{N}_{,j}.\bar{p}^j\right).dA\right\}\right] \quad (37)$$

Where

$$M^{ij} = \begin{bmatrix} M^{xx} & M^{xy} \\ M^{yx} & M^{yy} \end{bmatrix} \qquad (38)$$

$K_{EG} = \left\{\int_A N_{,i}.M^{ij}.N_{,j}.dA\right\}$: represents the combined action (bending and membrane stiffness matrix) of thin plate.

$R_2 = \left\{\int_A N_{,i}.M^{ij}.\left(\bar{N}_{,j}.\bar{p}^j\right).dA\right\}$: is the new term results from modified FEM.

M^{ij} : is the bending moments matrix.

The fourth term results from membrane forces in thin plate (after multiplying the brackets and applying the same notice)

$$T_4 = \frac{1}{2}.\left[\delta\bar{w}\left\{\int_A N_{,i}.S^{ij}.N_{,j}.dA\right\}.\bar{w} + \delta\bar{w}\left\{\int_A N_{,i}.S^{ij}.\left(\bar{N}_{,j}.\bar{P}^j\right).dA\right\}\right] \quad (39)$$

Where

$$S^{ij} = \begin{bmatrix} S^{xx} & S^{xy} \\ S^{yx} & S^{yy} \end{bmatrix} \qquad (40)$$

$K_G = \left\{\int_A N_{,i}.S^{ij}.N_{,j}.dA\right\}$: represents geometric stiffness matrix.

$R_3 = \left\{\int_A N_{,i}.S^{ij}.\left(\bar{N}_{,j}.\bar{P}^j\right).dA\right\}$: is the new term results from modified FEM.

S^{ij} : is the membrane forces matrix.

The right side of Eq (32) represents external forces work, substituting Eq (26) in it results

$$R = \int_A \tilde{q}_i\delta(\Delta w_i)dA = \left[\delta\bar{w}.\int_A \tilde{q}_i.N_i.dA\right] \qquad (41)$$

By substituting Eqs (35,37,39,41) into Eq (32) results

$$[K_T].\delta\bar{w} = R - R_1 - R_2 - R_3 \qquad (42)$$

Where

$$\left[K_E + K_{EG} + \frac{1}{2}.K_G\right] = [K_T] \qquad (43)$$

K_T : is the tangent stiffness matrix for geometric nonlinear analysis of thin plates.

Numerical Examples And Discussion

In this section, we will test and assess modified FEM through several geometrically nonlinear applications. Numerical results obtained by the present method are discussed and compared with those obtained from analytical and numerical solutions if available. In all examples, 40 equal incremental load steps are used, unless specified otherwise, and the convergence tolerance is taken to be 0.01.

A Clamped Thin Plate Subjected To Uniformly Distributed Load

A clamped thin square plate (A) subjected to a uniformly distributed load $q = 2 \times 10^4 psi$ is presented here. The thickness of the plate is $h = 1.0in$, the side length of the square plate is $a = b = 200in$, and the material properties are $E = 2 \times 10^{11} psi$, $\vartheta = 0.3$ as Fig 3 shows.

The calculated central deflection of the square plate obtained from the present method (modified FEM), FEM, MSC Patran [25] and the analytical solution [1,26] are shown in table1.

Method	Mesh	Central deflection (w , inch)	Difference (%)
Analytical [1,26]	10x10	1.2	-
FEM	10x10	1.2805	6.7%
MSC Patran [25]	10x10	1.2611	5%
Modified FEM	10x10	1.2258	2%

Table 1 : Central Deflection In Thin Clamped Square Plate (A)

It can be observed from the table 1 and Fig. 4 that, the central deflection results from modified FEM is closer to exact solution from other methods. The Relationship between central deflection and distributed load of thin clamped square plate is shown in Fig 4 for mesh (6×6) and (10×10). Also we draw the Relationship between nondimensional central deflection (w/h) and nondimensional distributed load $(q.a^4/E.h^4)$ for mesh (10×10) as shown in Fig. 5.

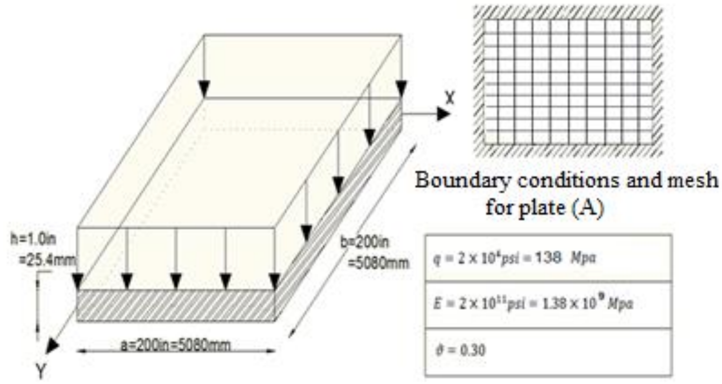


Fig. 3 : Clamped Thin Square Plate (A)

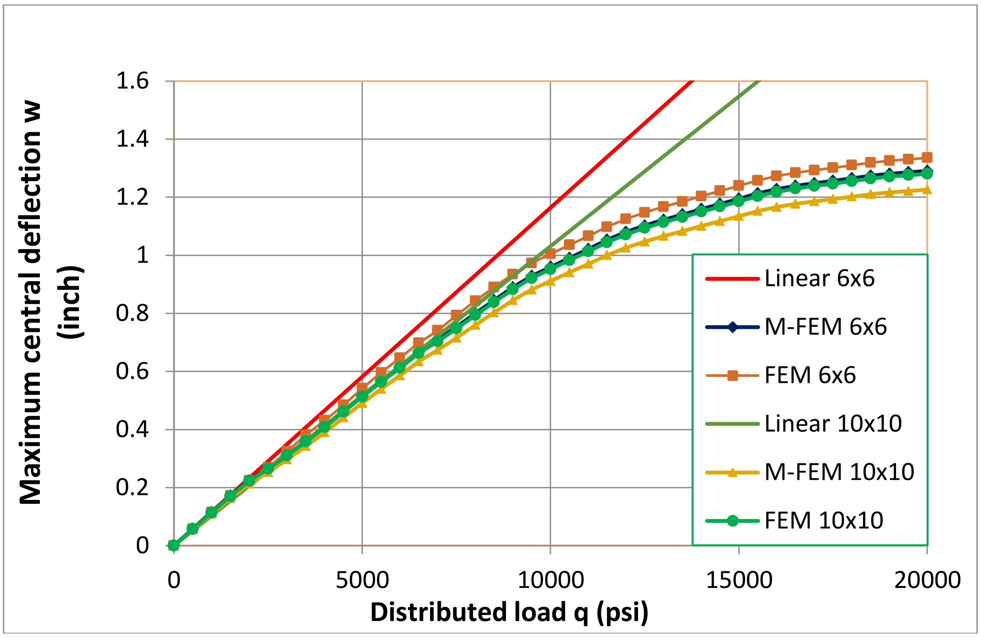


Fig 4 : Relationship Between Central Deflection And Distributed Load Of The Thin Clamped Square Plate (A)

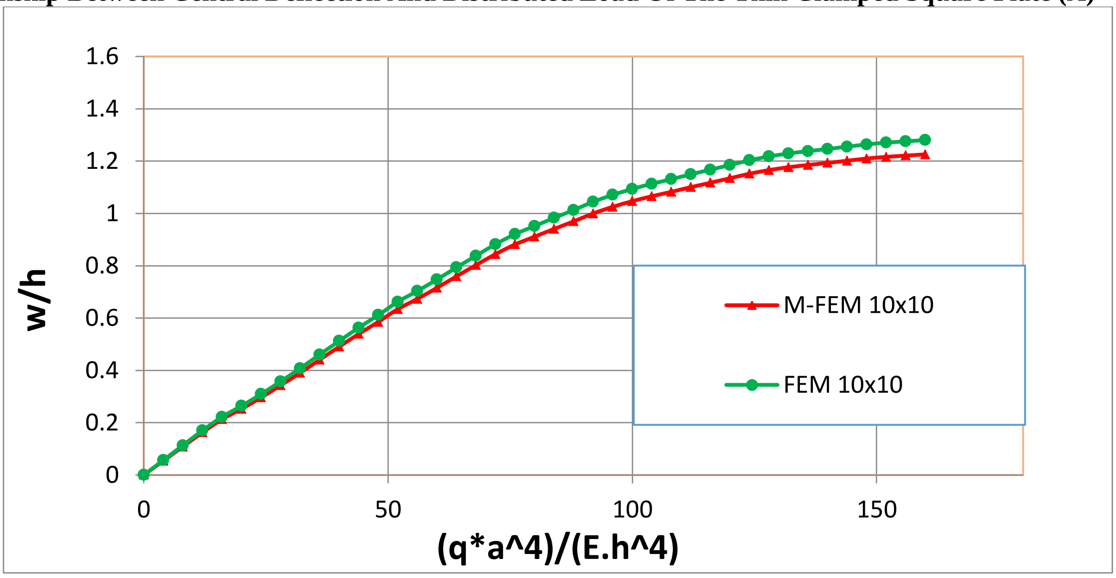


Fig 5 : Relationship Between Nondimensional Central Deflection $\left(\frac{w}{h}\right)$ And Nondimensional Distributed Load $\left(\frac{q.a^4}{E.h^4}\right)$ Of The Thin Clamped Square Plate (A)

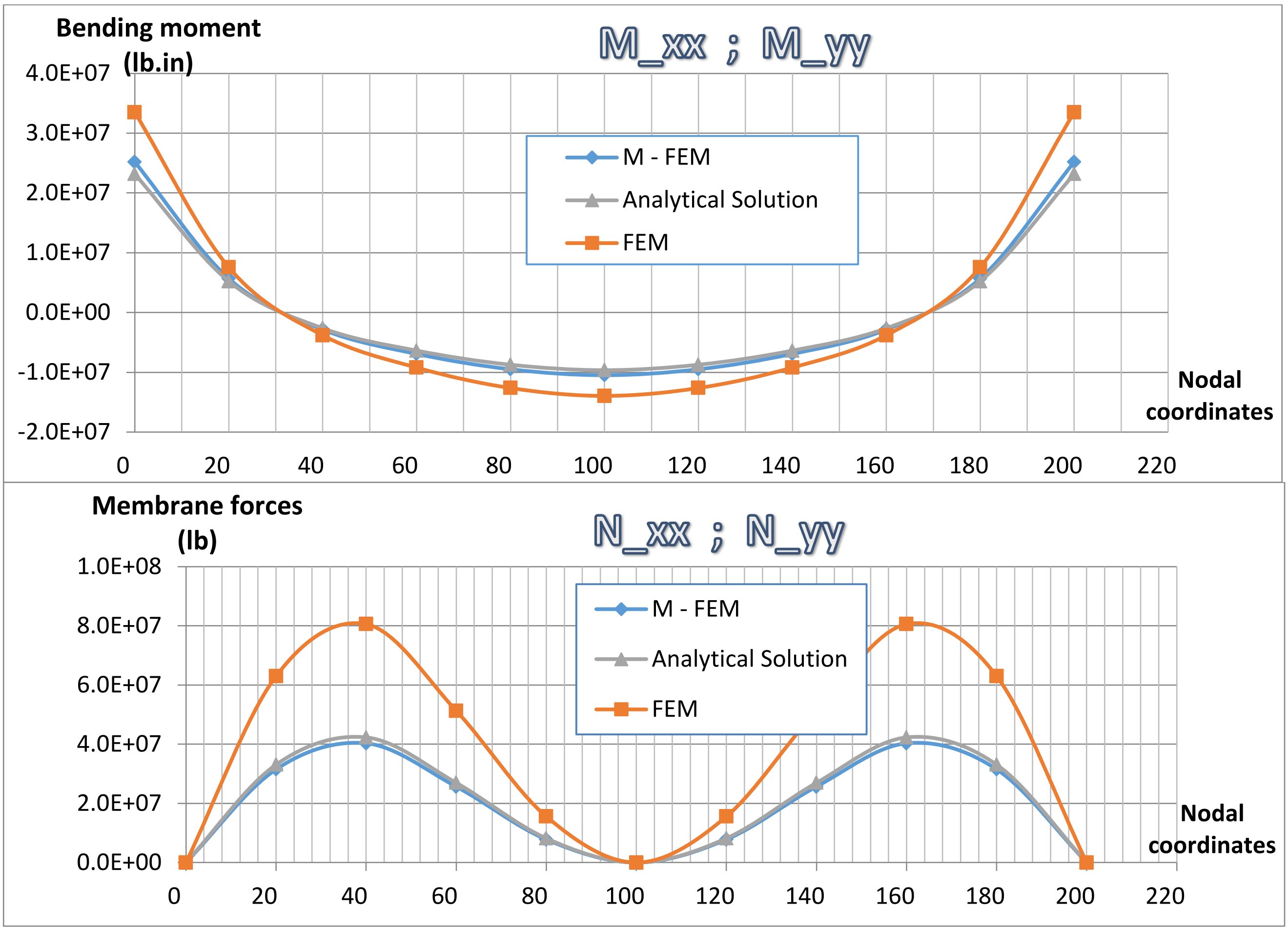


Fig 6 :Bending Moments$(M_{xx}\ ,\ M_{yy})$ And Membrane Forces $(N_{xx}\ ,\ N_{yy})$ Of Thin Clamped Square Plate (A) For Line Of Nodes At Center

The effect of modified FEM on internal forces (bending moments , membrane forces) of thin plates is shown in Fig 6, where the first diagram represents bending moments $(M_{xx}\ , M_{yy})$ for line of nodes located at coordinates $\left(x = \frac{a}{2}, y = 0 \rightarrow i * \frac{a}{10}\right)$ where $(i = 1 \rightarrow 10)$and the second one represents membrane forces$(N_{xx}\ ,\ N_{yy})$ for the same line of nodes.

It is obviously clear that internal forces results from modified method are in good agreement with analytical solution, while there is a difference between FEM and analytical solution. That is because modified FEM takes into account the differential equation of the problem when we form displacement function.

A Clamped – Simply Supported (Parallel Sides) Thin Plate From Aluminum Subjected To Uniformly Distributed Load

A clamped-simply supported thin square plate (B) subjected to a uniformly distributed load $q = 500psi$ is presented here. The thickness of the plate is $h = 1.5in$, the side length of the square plate is $a = b = 50in$, and the material properties are $E = 1.0 \times 10^7 psi$, $\vartheta = 0.3$ as Fig.7 shows.

The Relationship between central deflection and distributed load of thin clamped-simply supported square plate is shown in Fig 8 for mesh (10×10).

The calculated central deflection of the square plate obtained from the present method (modified FEM), FEM and the analytical solution [1,26] are shown in table 2.

Fig 9, represents bending moments $(M_{xx}\ ,\ M_{yy})$ for line of nodes located at coordinates $\left(x = \frac{a}{2}, y = 0 \to i * \frac{a}{10}\right)$ where $(i = 1 \to 10)$.

Fig 10, represents membrane forces$(N_{xx}\ ,\ N_{yy})$ for the same line of nodes.

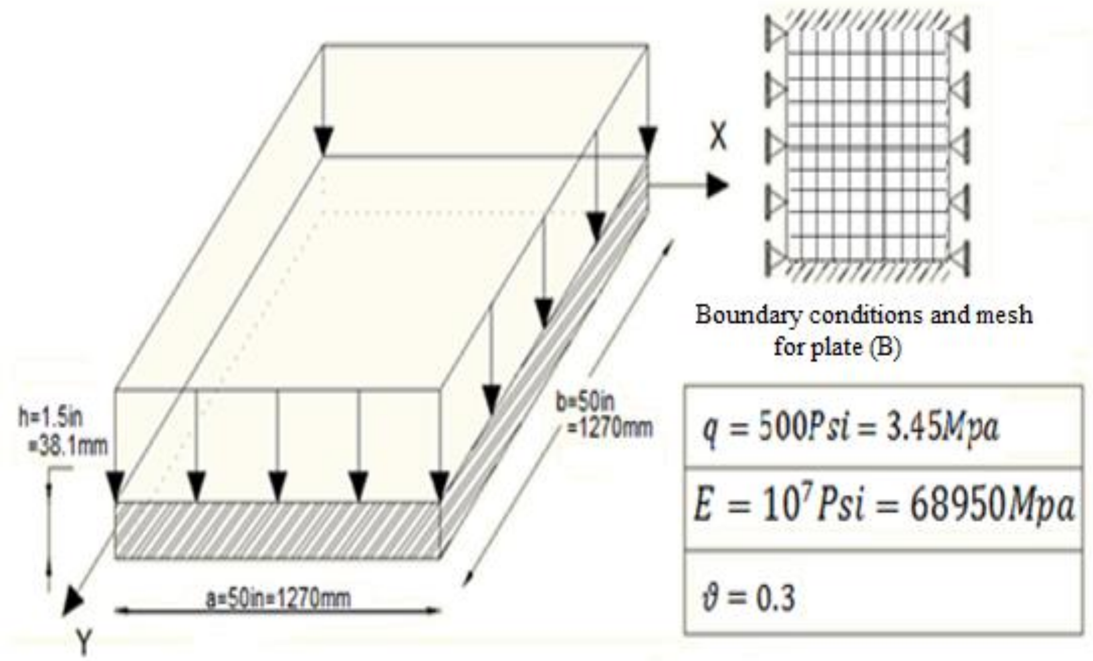


Fig. 7 : Clamped - Simply Supported Thin Square Plate (B)

Method	Mesh	Central deflection (w , inch)	Difference (%)
Analytical [1,26]	10x10	1.958	-
FEM	10x10	2.1144	8%
Modified FEM	10x10	2.0265	3.5%

Table 2 : Central Deflection In Thin Clamped - Simply Supported Square Plate (B)

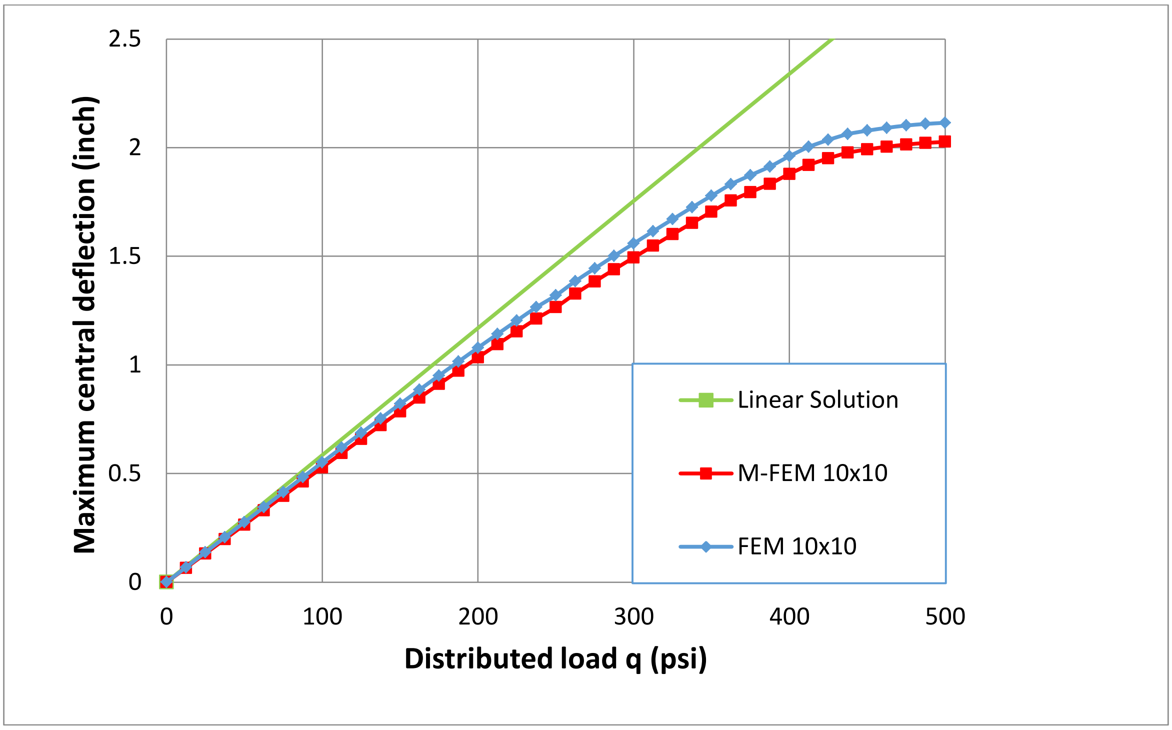
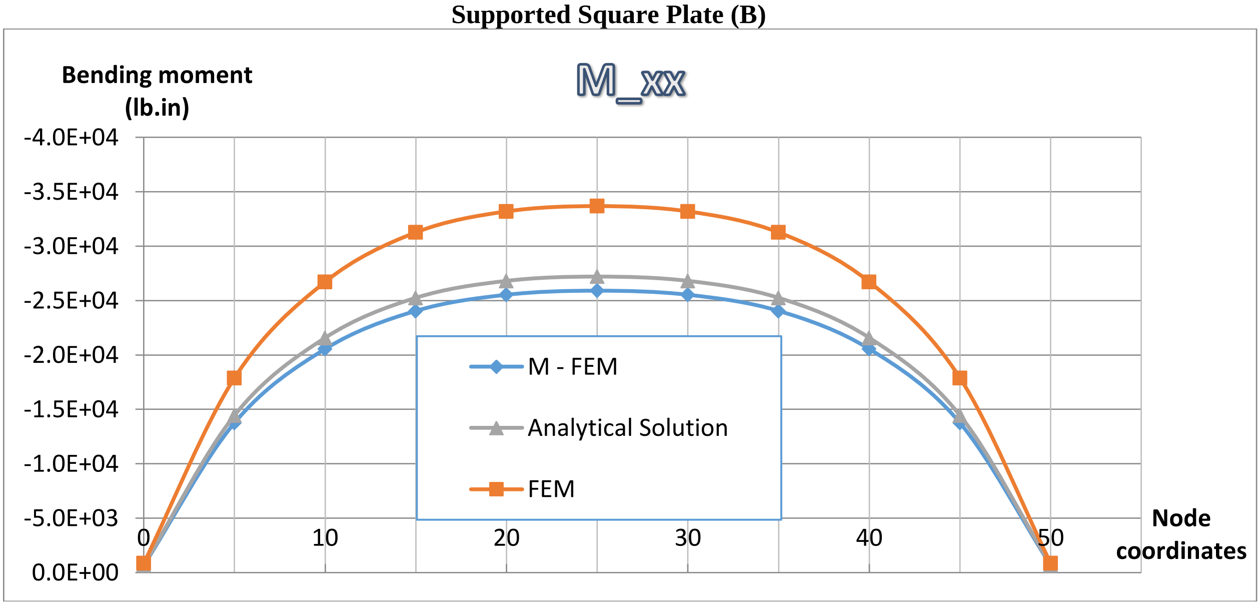
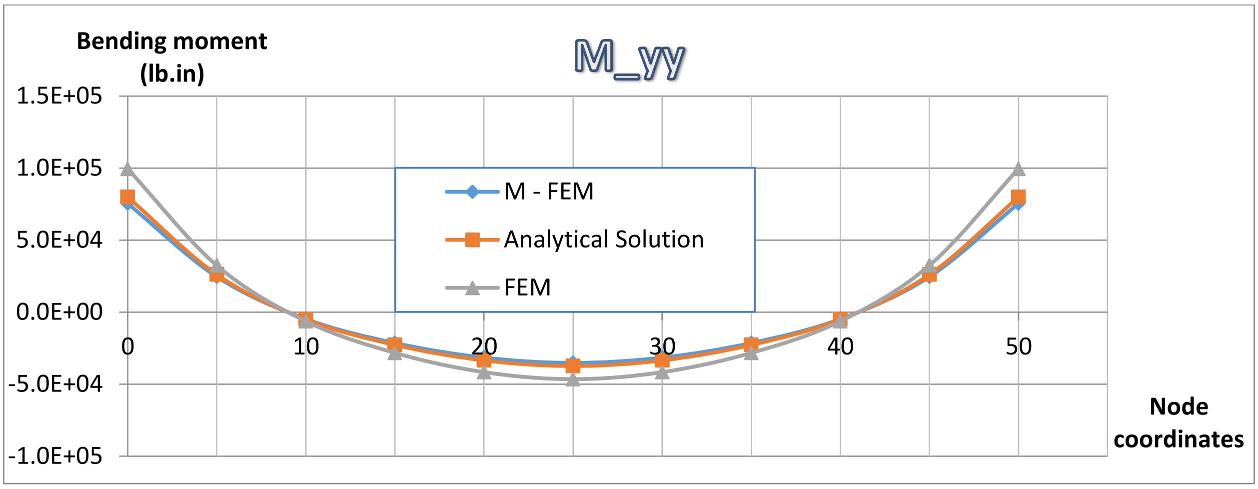


Fig 8 : Relationship Between Central Deflection And Distributed Load Of The Thin Clamped - Simply Supported Square Plate (B)

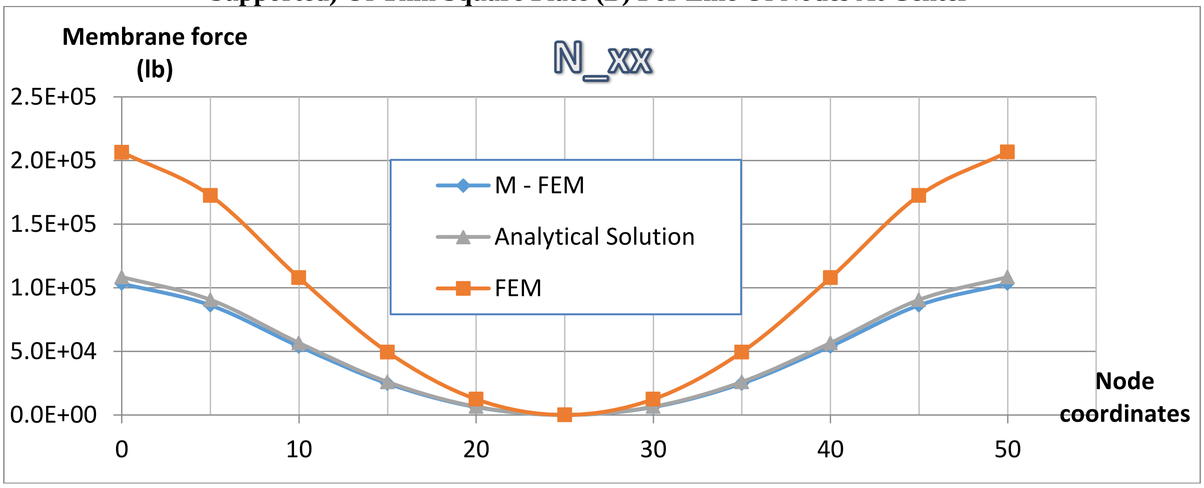


(a) : Bending Moment M_{xx}

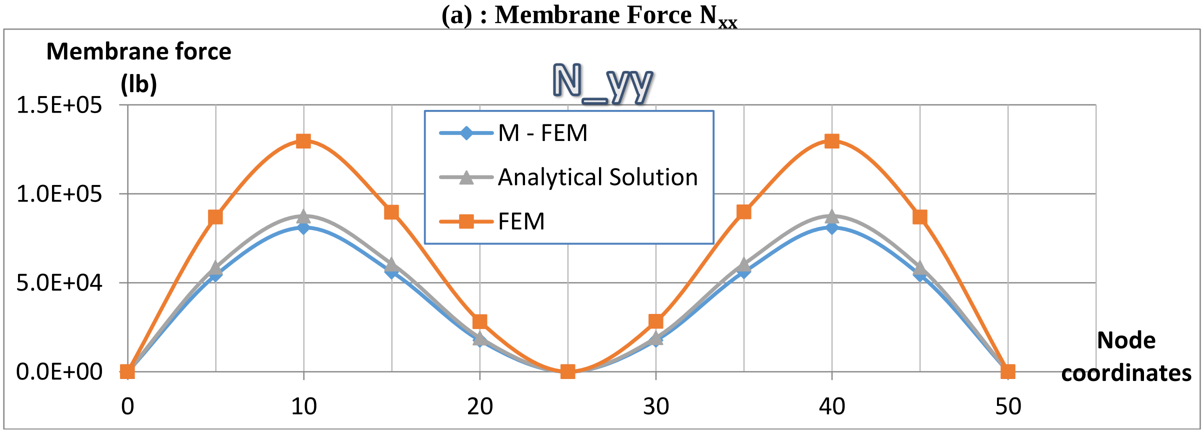


(b) : Bending Moment M_{yy}

Fig 9 : Bending Moments $(M_{xx}\ ,\ M_{yy})$ In Two Directions (Clamped - Clamped ; Supported - Supported) Of Thin Square Plate (B) For Line Of Nodes At Center



(a) : Membrane Force N_{xx}



(b) : Membrane Force N_{yy}

Fig 10 : Membrane Forces $(N_{xx}\ ,\ N_{yy})$ In Two Directions (Clamped - Clamped ; Supported - Supported) Of Thin Square Plate (B) For Line Of Nodes At Center

It can be observed from table 2 and Fig. 8 that, central deflection results from modified FEM is closer to exact solution (Analytical) from traditional FEM method.

Conclusions

1. By using the modified finite element method, the displacement function of traditional ACM element was separated into two parts (homogeneous, Non-homogeneous) for large deflection analysis of thin plates.
2. The results of the modified method were reasonable and in good agreement with the analytical solution especially for internal forces (bending moments – membrane forces).

Appendix A

$\overline{N}_{(1,1)} = \frac{-3.x^2.y^2(\vartheta^2-1)(2a^3.b^3 + a^3 + b^3)}{16.E.a^3.b^3.h^3}$	$\overline{N}_{(1,7)} = \frac{3.x^2.y^2(\vartheta^2-1)(-2a^3.b^3 + a^3 + b^3)}{16.E.a^3.b^3.h^3}$
$\overline{N}_{(1,2)} = \frac{-3x^2.y^2(b^3+1).(\vartheta^2-1)}{16.E.b^2.h^3}$	$\overline{N}_{(1,8)} = \frac{3x^2.y^2(b^3-1).(\vartheta^2-1)}{16.E.b^2.h^3}$
$\overline{N}_{(1,3)} = \frac{3x^2.y^2(\vartheta^2-1).(a^3-a+1)}{16.E.a^2.h^3}$	$\overline{N}_{(1,9)} = \frac{3x^2.y^2(\vartheta^2-1).(-a^3+a+1)}{16.E.a^2.h^3}$
$\overline{N}_{(1,4)} = \frac{-3.x^2.y^2(\vartheta^2-1)(2a^3.b^3 + a^3 - b^3)}{16.E.a^3.b^3.h^3}$	$\overline{N}_{(1,10)} = \frac{-3.x^2.y^2(\vartheta^2-1)(2a^3.b^3 - a^3 + b^3)}{16.E.a^3.b^3.h^3}$
$\overline{N}_{(1,5)} = \frac{-3x^2.y^2(b^3+1).(\vartheta^2-1)}{16.E.b^2.h^3}$	$\overline{N}_{(1,11)} = \frac{3x^2.y^2(b^3-1).(\vartheta^2-1)}{16.E.b^2.h^3}$
$\overline{N}_{(1,6)} = \frac{3x^2.y^2(\vartheta^2-1).(-a^3+a+1)}{16.E.a^2.h^3}$	$\overline{N}_{(1,12)} = \frac{3x^2.y^2(\vartheta^2-1).(a^3-a+1)}{16.E.a^2.h^3}$

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