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دراسة لبعض المسائل في نظرية الحلقات النتروسوفية

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□ ملخّص □

الهدف من هذا البحث هو دراسة بعض المسائل في البنى الجبرية النتروسوفية .

كما نقدم بعض المسائل البحثية المفتوحة عن هذه البنى الجبرية النتروسوفية وتحديدا الحلقات النتروسوفية، الحلقات النتروسوفية المصفاة والحلقات النتروسوفية المصفاة من المرتبة n .

كما نتعرض أيضا لمفهوم حلقة الأعداد النتروسوفية العقدية مع بعض المسائل البحثية المفتوحة المرتبطة بها والتي تدعى مسائل سمرنداكة–كانداسمي حول خصائص هذه الأعداد وبالتحديد الجبرية منها .

الكلمات المفتاحية: حلقة نتروسوفية–حلقة نتروسوفية مصفاة–حلقة نتروسوفية من المرتبةn– أعداد_عقدية نتروسوفية مثاليات AH

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Neutrosophic Ring Theory A Study On Some Interesting Questions In

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□ABSTRACT □

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Abstract: The aim of this paper is to present a study about recent progressions in the study of neutrosophic algebraic structures. Also, it lists the most interesting open questions about neutrosophic algebraic structures such as neutrosophic rings, refined neutrosophic rings and *n*-refined neutrosophic rings.

On the other hand, a study on neutrosophic ring of complex numbers has been presented, with many Smarandache-Kandasamy open problems about the properties of these numbers, especially algebraic ones.

Key words: Neutrosophic ring, refined neutrosophic ring, *n*-refined neutrosophic ring, neutrosophic complex number, AH-ideals.

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1. Introduction

Neutrosophy is a new kind of logic proposed by Smarandache to generalize the intuitionistic fuzzy logic [1], where he defined the concept of neutrosophic sets. These sets played a basic role in different field of applied mathematics such as pattern recognition [2], machine learning [3], health care [4], computer science [5], and statistics [6]. Other applications of neutrosophy can be found in [29,30,35,37,40-43,45,47,50].

In the last few years, neutrosophic sets became a useful tool in the study of pure mathematics like topology [7], algebraic equations [8], number theory [9], and it was used in the study of algebraic structures such as modules [10,11], spaces [12,13], and groups [14,15].

The concept of neutrosophic ring was defined firstly by Kandasamy and Smarandache in [16], and then it was studied by Agboola et.al in [17]. Recently, many generalizations were defined and handled such as refined neutrosophic rings [18,19], and n-refined neutrosophic rings [20].

These rings were studied widely, where Kandasamy et.al determined the form of triplets group [21], idempotents [22], and semi idempotents [23]. Recently, many researchers concentrate their efforts to study these structures, where AH-substructures were defined in [24,25], refined idempotents [26], and AH-homomorphisms [27].

An important generalization was presented in [28] by Smarandache and Kandasamy, where they defined neutrosophic complex numbers ring, and neutrosophic finite complex numbers.

Through this paper, we study the basic definitions and theorems about neutrosophic rings, refined neutrosophic rings, and *n*-refined neutrosophic rings. In addition, we study some of important open questions about those rings. These questions are still open, their answers will have a huge effect in theoretical progression in many related fields. Also, we add some examples to ensure and clarify these concepts in general.

2. Neutrosophic rings

Definition 2.1: [24]

Let R be a ring, I be the indeterminacy with property $I^2 = I$, then the neutrosophic ring is defined as follows:

$R(I) = \{a + bI; a, b \in R\}$.

Definition 2.2: [24]

Let *R(I)* be a neutrosophic ring, it is called commutative if and only if $xy = yx \ \forall\, x, y \in R(I)$.

Definition 2.3: [17]

(a) Let x be any element in R(I), it is called idempotent if and only if $x^2 = x$.

(b) x, y are called a duplet if and only if $xy = yx = x$.

Definition 2.4: [17]

Let *R(I)* be a neutrosophic ring, a non-empty subset P of R(I) is called a neutrosophic ideal if

.)a) P is a neutrosophic subring of R(I(

(b) $rx \in P \ for\ every\ x \in P\ and\ r \in R(I)$.

Definition 2.5:[24]
Let *R(I)* be a neutrosophic ring and $P = P_0 + P_1 I = \{a_0 + a_1 I\ ;\ a_0 \in P_0\ , a_1 \in P_1\}$.
(a)We say that P is an AH-ideal if P_0, P_1 are ideals in the ring R.
(b)We say that P is an AHS-ideal if $P_0 = P_1$.
(c) *T*he AH-ideal P is called null if $P_0, P_1 \in \{R\ , O\ \}$.
Theorem 2.1:[24]
Let *R(I)* be a neutrosophic ring and $P = P_0 + P_1 I$ be an AH-ideal, then P is not a neutrosophic ideal in general by the classical meaning.
Definition 2.6:[24]
Let *R(I)* be a neutrosophic ring and $P = P_0 + P_1 I$ be an AH-ideal, we define the AH-factor as: $R(I)/P = R/P_0 + R/P_1\ I$.
Definition 2.7:[24]
Let *R(I)* be a neutrosophic ring and $P = P_0 + P_1 I$ be an AH-ideal then $R(I)/P$ is a ring with the following two binary operations:
$[(x_0 + P_0) + (y_0 + P_1)I] + [(x_1 + P_0) + (y_1 + P_1)I] = [(x_0 + x_1 + P_0) + (y_0 + y_1 + P_1)I]$.
$[(x_0 + P_0) + (y_0 + P_1)I] \times [(x_1 + P_0) + (y_1 + P_1)I] = [(x_0 \times x_1 + P_0) + (y_0 \times y_1 + P_1)I]$.
Definition 2.8:[24]
Let *R(I)*, *T(J)* be two neutrosophic rings and the map $f: R(I) \to T(J)$ we say that f is a neutrosophic AHS-homomorphism if:
The restriction of the map f on R is a ring homomorphism from R to T i.e $f_R: R \to T$ is homomorphism and
$f(a + b\mathrm{I}) = f_R(a) + f_R(b)J$.
We say that *R(I)*, *T(J)* are AHS-isomorphic neutrosophic rings if there is a neutrosophic AHS-homomorphism
$f: R(I) \to T(J)$ which is a bijective map i.e (R $\cong T$), we say that f is a neutrosophic AHS-isomorphism.
Theorem 2.2:[24]
Let *R(I)*, *T(J)* be two neutrosophic rings and *f: R(I)→T(J)* is a neutrosophic ring AHS-homomorphism, let $P = P_0 + P_1 I$ be an AH-ideal of R(I) and $Q = Q_0 + Q_1 J$ be an AH-ideal of *T(J)*, then we have:
(a) *f(P)* is an AH-ideal of *f(R(I))*.
(b) $f^{-1}(Q)$ is an AH-ideal of *R(I)*.
(c) If *P* is an AHS-ideal of *R(I)*,then *f(P)* is an AHS-ideal of *f(R(I))*.
(d) $AH - kerf = kerf_R + kerf_R I$ is an AHS-ideal;f_R is the restriction of *f* on the ring *R*.
(e) The AH-factor $R(I)/kerf\ \ is\ AHS - isomorphic\ to\ f(R(I))$.
Example 2.1:
Let R(I)=$Z_6(I)$, $P_0 = \{0,2,4\}$, $P_1 = \{0,3\}$are two ideals in Z_6then we have:
(a) P=$P_0 + P_1 I$ = {0,2,4 ,2+3I,4+3I, 3I} is an AH-ideal.
(b) Q=$P_1 + P_1 I = \{0,3,3 + 3I, 3I\}$ is an AHS-ideal because $P_1 = P_1$.
(c) We have: R/P_0={ P_0 , 1+P_0 } and $R/P_1 = \{P_1\ , 1 + P_1\ , 2 + P_1\}$; thus the AH-factor
$R(I)/P$ = { $P_0 + P_1 I$, $P_0 + (1 + P_1)I$, $P_0 + (2 + P_1)I$, (1+P_0) + $P_1 I$, (1+P_0) + $(1 + P_1)I$, (1+P_0) + $(2 + P_1)I$ }
We should Remark that $P_0 = P_0 + 0.I$ and 0 = 0 + 0.I.
(d)We can clarify the addition on the AH-factor $R(I)/P$ as:

$[P_0 + (1 + P_1)I] + [(1 + P_0) + (2 + P_1)I$]= [(0+1)+P_0] + [(1+2)+P_1] I = (1+P_0) + (3+P_1) I = (1+P_0) + $P_1 I$.

We can clarify the multiplication on the AH-factor $R(I)/P$ as:

[$P_0 + (1 + P_1)I] \times [(1 + P_0) + (2 + P_1)I] = [(0 \times 1) + P_0] + [(1 \times 2) + P_1]I = P_0 + (2 + P_1)I$.

(e) We can see that $P \cap Q = (P_0 \cap P_1) + (P_1 \cap P_1)I = \{0\} + P_1 I = \{0,3I\}$ which it is an AH-ideal.

(f) P+Q = $(P_0 + P_1) + (P_1 + P_1)I = R + P_1 I = \{0,1,2,3,4,5,3I, 1 + 3I, 2 + 3I, \ldots \ldots 5 + 3I\}$.

Open problem (1): Describe the algebraic structure of the group of units of a neutrosophic ring R(I)?. What is its relation with the group of units of the ring R?.

3. Refined neutrosophic rings

Definition 3.1:[44]

The element I can be split into two indeterminacies I_1 , I_2 with conditions:

$I_1{}^2 = I_1$, $I_2{}^2 = I_2$, $I_1 I_2 = I_2 I_1 = I_1$.

Definition 3.2:[44]

If X is a set then $X(I_1, I_2) = \{(a, bI_1, cI_2): a , b , c \in X \}$ is called the refined neutrosophic set generated by X , I_1, I_2.

Definition 3.3:[44]

Let *(R,+,×)* be a ring, *(R(I_1, I_2) , + ,×)* is called a 2-refined neutrosophic ring generated by R ,I_1, I_2.

Example 3.1:

The refined neutrosophic ring of integers is $Z(I_1, I_2)$={$(a, bI_1, cI_2); a, b, c \in Z$}.

Definition 3.4:[25]

Let *(R(I_1, I_2) , + , .)* be a refined neutrosophic ring and P_0, P_1, P_2 be ideals in the ring R then the set $P = (P_0, P_1 I_1, P_2 I_2) = \{(a, bI_1, cI_2): a \in P_0, b \in P_1, c \in P_2\}$ is called a refined neutrosophic AH-ideal.

If $P_0 = P_1 = P_2$ then P is called a refined neutrosophic AHS-ideal.

Definition 3.5:[25]

(a) Let f: $R(I_1, I_2) \to T(I_1, I_2)$ be an AHS-homomorphism we define AH-Kernel of f by : $AH - Kerf = \{ (a, bI_1, cI_2); a , b , c \in Kerf_R \} = (Kerf_R , Kerf_R I_1 , Kerf_R I_2)$

(b) let S=($S_0 , S_1 I_1, S_2 I_2$) be a subset of $R(I_1, I_2)$, then : $f(S) = (f_R(S_0), f_R(S_1)I_1, f_R(S_2)I_2) = \{(f_R(a_0), f_R(a_1)I_1, f_R(a_2)I_2); a_i \in S_i\}$

(c) let S=(S_0 ,$S_1 I_1, S_2 I_2$) be a subset of $T(I_1, I_2)$, then :

$f^{-1}(S) = (f_T^{-1}(S_0), f_T^{-1}(S_1)I_1, f_T^{-1}(S_2)I_2)$.

Theorem 3.1:[19]

Let *(R ,+,×)* be a ring and *R(I)*, *R(I_1, I_2)* the related neutrosophic ring and refined neutrosophic ring respectively, we have:

(a) There is a ring homomorphism $f: R(I_1, I_2) \to R(I)$.

(b) The additive group (*Ker f* , *+)* is isomorphic to the additive group *(R,+)*.

Theorem 3.2:[19]

Let R be a ring, where *Char(R) = 2*, there is a subring of $R(I_1, I_2)$ *say K* with property $K \cong R$; $R(I_1, I_2)/K \cong R(I)$.

Example 3.2:

Let R = Z_2, R is a ring with respect to addition and multiplication modulo 2, we have *Char(R) =2.*

$R(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in Z_2\}$ is the related refined neutrosophic ring.

$R(I) = \{a + bI ; a, b \in Z_2\}$ is the related neutrosophic ring.

Let f be the ring homomorphism defined above, we have $= Ker\ f = \{(0, bI_1, -bI_2); b \in Z_2\} = \{(0,0,0), (0, I_1, I_2)\} \cong R$.

$R(I_1, I_2)/K = \{ K, (1,0,0) + K, (0, I_1, 0) + K, (1,0, I_2) + K \} \cong R(I)$.

$O(R(I_1, I_2)) = 8, O(R(I)) = 4, O(R) = 2$, we find $O(R(I_1, I_2))/O(R) = O(R(I)) = 4$.

Theorem 3.3:[19]

Let R be a ring, there is a subring of $R(I)$ $say\ K$ with property $K \cong R$; $R(I)/K \cong R$.

Definition 3.6:[25]

Let $(R(I_1, I_2), +, .)$ be a refined neutrosophic ring and $P = (P_0, P_1I_1, P_2I_2)$ be an AH-ideal then :

(a) We say that P is a weak prime AH-ideal if $P_i; i \in \{0,1,2\}$ are prime ideals in R.

(b) We say that P is a weak maximal AH-ideal if $P_i; i \in \{0,1,2\}$ are maximal ideals in R.

(c) We say that P is a weak principal AH-ideal if $P_i; i \in \{0,1,2\}$ are principal ideals in R.

(d) We define AH-factor as : $R(I_1, I_2)/P = (R/P_0, R/P_1\ I_1, R/P_0\ I_2)$.

Example 3.3:

Let $R = (Z_6, +, .)$, $T = (Z_{10}, +, .)$ be two rings and f is the AHS-homomorphism defined, we have the following :

(a) $P_0 = \{0,2,4\}, P_1 = \{0,3\}$ are two ideals in Z_6 thus $P = (P_0, P_0I_1, P_1I_2)$ is an AH-ideal of $R(I_1, I_2)$

(b) $f(P) = (f(P_0), f(P_0)I_1, f(P_1)I_2) = \{ (0,0,0), (0,0,5I_2) \}$ is a refined neutrosophic AH-ideal in $T(I_1, I_2)$.

(c) $Q_0 = \{0,2,4,6,8\}$ is a maximal ideal in Z_{10} and $f_T^{-1}(Q_0) = \{0,2,4\}$, so $Q = (Q_0, Q_0I_1, Q_0I_2)$ is a weak maximal refined neutrosophic AHS-ideal in $T(I_1, I_2)$, we have $f^{-1}(Q) = (\{0,2,4\}, \{0,2,4\} I_1, \{0,2,4\} I_2)$ is a weak maximal refined neutrosophic AHS-ideal in $R(I_1, I_2)$.

Example 3.4:

(a) In the ring *(Z,+, .)* , P= <3> , Q=<2> are two prime and maximal ideals, thus $M=(P, , QI_1, QI_2) = \{(3a, 2bI_1, 2cI_2) ; a, b, c \in Z\}$ is a weak maximal/prime refined neutrosophic AH-ideal of $(Z(I_1, I_2), +, .)$.

(b)The map $f_Z : Z \to Z_6 ; f(a) = a\ mod\ 6$ is a homomorphism so the related refined neutrosophic AHS-homomorphism is

$f: Z(I_1, I_2) \to Z_6(I_1, I_2) ; f(a, bI_1, cI_2) = (a\ mod6, (b\ mod6)I_1, (c\ mod6)I_2)$, AH-ker f = $(6Z, 6ZI_1, 6ZI_2) \leq M$ since $6Z \leq P, Q$.

(c) $f(M) = (\{0,3\}, \{0,2,4\}I_1, \{0,2,4\}I_2)$ is a weak maximal/prime refined neutrosophic AH-ideal of $Z_6(I_1, I_2)$ **.**

Theorem 3.4:[25]

Let $f: R(I_1, I_2) \to T(I_1, I_2)$ be an AHS-homomorphism then we have :

(a)AH-Ker(f) is an AHS-ideal of $R(I_1, I_2)$.

(b) If P is a refined neutrosophic AH-ideal of $R(I_1, I_2)$, then $f(P)$ is a refined neutrosophic AH-ideal of $T(I_1, I_2)$.

(c) If P is a refined neutrosophic AHS-ideal of R(I_1, I_2) , then *f(P)* is a refined neutrosophic AHS-ideal of $T(I_1, I_2)$.

Open problem (2): Describe the algebra**ic** structure of the group of units of any refined neutrosophic ring?

Open problem (3): Determine the form of primes in the refined neutrosophic ring of integers.

Open problem (4): Is Euler's Theorem still true in the refined neutrosophic ring of integers?.

Open problem (5): Is the fundamental Theorem of arithmetic still true in the refined neutrosophic ring of integers?.

4. *n*-Refined neutrosophic rings

Definition 4.1:[20]

Let *(R,+,×)* be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n*(I)*=$\{a_0 + a_1 I + \cdots + a_n I_n \ ; \ a_i \in R\}$ to be *n*-refined neutrosophic ring. If *n=2* we get a ring which is isomorphic to 2-refined neutrosophic ring $R(I_1, I_2)$, then

Addition and multiplication on R_n*(I)* are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i \, , \sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n} (x_i \times y_j) I_i I_j,$

where × is the multiplication defined on the ring *R*.

It is easy to see that R_n(I) is a ring in the classical concept and contains a proper ring *R*.

Definition 4.2:[20]

Let R_n(I) be an *n*-refined neutrosophic ring, it is said to be commutative if $xy = yx$ for each x , $y \in R_n$(I), if there is 1∈ R_n(I) such $1.x = x.1 = x$, then it is called an *n*-refined neutrosophic ring with unity.

Definition 4.3:[20]

(a) Let R_n*(I)* be an *n*-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i = \{ a_0 + a_1 I + \cdots + a_n I_n : a_i \in P_i\}$ where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of *R* for all i, AHS-subring is defined by the condition $P_i = P_j$ for all i , j.

(b) *P* is an AH-ideal if P_i is an two sides ideal of *R* for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i , j.

(c) The AH-ideal *P* is said to be null if $P_i = R \ or \ P_i = \{0\}$ for all *i*.

Theorem 4.1:[20]

Let R_n*(I)* be an *n*-refined neutrosophic ring and *P* is an AH-ideal, *(P,+)* is an abelian neutrosophic group with $k \leq n$ and r.p∈ P for all $p \in P$ and $r \in R$.

Theorem 4.2:[20]

Let R_n(I) be an *n*-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i$, $Q = \sum_{i=0}^{n} Q_i I_i$ be two AH-ideals then P+Q, P∩Q are AH-ideals. If P, Q are AHS-ideals then P+Q, P∩Q are AHS-ideals.

Theorem 4.3:[20]

Let R_n(I) be an *n*-refined neutrosophic ring and $P = \sum_{i=0}^{n} P_i I_i$ be an AH-ideal, we define AH-factor *R(I)/P* = $\sum_{i=0}^{n} (R/P_i) I_i = \sum_{i=0}^{n} (x_i + P_i) I_i \, ; x_i \in R$.

Definition 4.4:[20]

a) Let R_n(I), T_n(I) be two *n*-refined neutrosophic rings respectively, and $f_R: R \to T$ be a ring homomorphism. We define *n*-refined neutrosophic AHS-homomorphism as :

f: R_n(I)→ T_n(I); $f(\sum_{i=0}^{n} x_i I_i) = \sum_{i=0}^{n} f_R(x_i) I_i$.

(b) f is an n-refined neutrosophic AHS-isomorphism if it is a bijective n-refined neutrosophic AHS-homomorphism.

(c) AH-Ker f = $\sum_{i=0}^{n} Ker(f_R) I_i$= {$\sum_{i=0}^{n} x_i I_i$; $x_i \in Ker\ f_R$}.

Theorem 4.4:[20]

Let R_n(I), T_n(I) be two n-refined neutrosophic rings respectively and f be an n-refined neutrosophic AHS-homomorphism f: R_n(I)→ T_n(I). Then

(a) If P = $\sum_{i=0}^{n} P_i I_i$is an AH- subring of R_n(I) then *f(P)* is an AH- subring of T_n(I),

(b) If P = $\sum_{i=0}^{n} P_i I_i$ is an AHS- subring of R_n(I) then *f(P)* is an AHS- subring of T_n(I),

(c) If P = $\sum_{i=0}^{n} P_i I_i$ is an AH-ideal of R_n(I) then *f(P)* is an AH-ideal of f(R_n(I)),

(d) P = $\sum_{i=0}^{n} P_i I_i$ is an AHS-ideal of R_n(I) then *f(P)* is an AHS-ideal of f(R_n(I)),

(e) R_n(I)$/AH - Ker(f)\ is\ AHS - isomorphic\ to\ f(R(I))$,

(f) Inverse image of an AH-ideal P in T_n(I) is an AH-ideal in R(I).

Example 4.1:

Let $R = Z_8$ be a ring with addition and multiplication modulo 8.

(a) 3-refined neutrosophic ring related with R is Z_{8_3}(I)={a+$bI_1 + cI_2 + dI_3; a, b, c, d \in Z_8$}.

(b) P={0,4} is an ideal of R, $\sqrt{P}$={0,2,4,6}, $M = P + PI_1 + PI_2 + PI_3$ is an AHS-ideal of Z_{8_3}(I),

$AH - Rad(M) = \sqrt{P} + \sqrt{P}I_1 + \sqrt{P}I_2 + \sqrt{P}I_3$which is an AHS-ideal of Z_{8_3}(I).

Example 4.2:

Let R= Z_2 the ring of integers modulo 2, let $n = 3$. The corresponding 3-refined neutrosophic ring is

Z_{2_3} (I)={0,1, $I_1, I_2, I_3, 1 + I_1, 1 + I_2, 1 + I_3, I_1 + I_2, I_1 + I_3, I_1 + I_2 + I_3, I_2 + I_3, 1 + I_1 + I_2 + I_3, 1 + I_2 + I_3, 1 + I_1 + I_3, 1 + I_1 + I_2$}.

Open problem (6): Describe the algebraic structure of the group of units in any *n*-refined neutrosophic ring.

Open problem (7): Determine the necessary and sufficient condition for the idempotency in any *n*-refined neutrosophic ring.

Open problem (8): Define AH-homomorphisms, do they have an *n*-refined neutrosophic ring structure?

Open problem (9): Define number theoretical concepts over the *n*-refined neutrosophic ring of integers.

Open problem (10): Determine an algorithm to solve Diophantine equations (especially Pell's equation) in the *n*-refined neutrosophic ring of integers.

Open problem (11): Determine the form of primes, Euler's Theorem in the *n*-refined neutrosophic ring of integers.

5. Neutrosophic complex numbers rings and fields

Definition 5.1:[28]

Let R be the field of real numbers, neutrosophic complex numbers set is defined as

$C(R \cup I) = \{a + bI + ci + dIi; a, b, c, d \in R\}$.

Definition 5.2:[28]
Let Q be the field of rational numbers, rational neutrosophic complex numbers set is defined as
$C(Q \cup I) = \{a + bI + ci + dIi; a, b, c, d \in Q\}$.
Definition 5.3:[28]
Let Z be the ring of integer numbers, integer neutrosophic complex numbers set is defined as
$C(Z \cup I) = \{a + bI + ci + dIi; a, b, c, d \in Z\}$.
Definition 5.4:[28]
Let Z_n be the ring of integers modulo *n*, the finite ring of complex numbers is defined as
$C(Z_n) = \{a + bi;\ i^2 \equiv -1(mod\ n)\}$.
Definition 5.5:[28]
We define the neutrosophic ring of finite complex numbers as follows:
$C(Z_n \cup I) = \{a + bi + cI + diI;\ i^2 \equiv -1(mod\ n)\}$.
Remark 5.1:[28]
Neutrosophic complex numbers is considered as a generalization of classical complex numbers.
Example 5.1:
Let Z_3 be the ring of integers modulo 3, then
$C(Z_3) = \{a + bi;\ i^2 \equiv -1\ (mod\ 3)\} = \{0,1,2, i, 1 + i, 2 + i, 2i, 1 + 2i, 2 + 2i\}$.
Example 5.2:
$C(Z_3 \cup I) = \{a + bi + cI + diI; a, b, c \in \{0,1,2\}\}$.
Open problem (12): Can $C(< Z \cup I >)$ be a Smarandache ring?
Open problem (13): Can $C(< R \cup I >)$ have irreducible polynomials?.
Open problem (14): Determine the irreducible polynomials over $C(< Q \cup I >)$?.
Open problem (15): Find irreducible polynomials in $C(< Z \cup I >)[x]$?. Is every ideal in $C(< Z \cup I >)$ is principal?.
Open problem (16): Can one say for all polynomials with complex neutrosophic coefficients $C(\langle R \cup I\rangle)$ is algebraically closed?.
Open problem (17): Describe the group of units structure in $C(< Z \cup I >), C(< R \cup I >), C(< Q \cup I >)$.
Open problem (18): Find zero divisors and units in $C(Z_{24})$. In general, determine the sufficient and necessary condition to any element to be a zero divisor.
Open problem (19): Find the form of ideals and maximal ideals in $C(\langle Z_n \cup I\rangle)$.
Open problem (20): Find a necessary and sufficient condition for a complex modulo integers ring
$S = C(Z_n)$ to have ideal I such that $C(Z_n)/I$ is never a field.

Open problem (21): Determine the algebraic structure of the group of units of the ring $C(\langle Z_n \cup I\rangle)$.
Open problem (22): Find the relationship between algebraic elements in any neutrosophic ring and its corresponding classical ring.
Open Problem (23): Find the relationship between algebraic elements in any refined neutrosophic ring and its corresponding classical ring.
Open Problem (24): Find the relationship between algebraic elements in any *n*-refined neutrosophic ring and its corresponding classical ring.

Open Problem (25): Describe the algebraic structure of the group of units in the ring $C(Z_n)$.

6. Conclusion

In this article, we have recalled some important and new trends in algebra, where the concept of neutrosophic ring, refined neutrosophic ring, *n*-refined neutrosophic ring, and neutrosophic complex numbers' ring are discussed.

Also, we have presented a review about the most interesting 25 open problems about these concepts. Many of these questions were asked by Smarandache and Kandasamy in [28] about the structure of neutrosophic complex ring numbers. These open questions will give a wide push to theoretical studies if they have been solved.

We presented many interesting examples to clarify the validity of these concepts and to clarify their structures.

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