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دراسة الأنظمة الديناميكية المستمرة في فضاءات هيلبرت وتطبيقاتها المرتبطة بمؤثرين مضطرد أعظمي وكوكيرسيف

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□ ملخّص □

في هذا المقال، نقترح دراسة فئة من الأنظمة الديناميكية المستمرة في فضاء هيلبرت ومصممة لحل مشاكل الأمثليات المرتبطة بمجموع مؤثرين: مؤثر عام مضطرد ومؤثر كوكويرسيف.

الهدف من هذا العمل تطوير ديناميكيات نيوتن المنتظمة التي تم تقديمها في مجلة نظرية وتطبيقات في مجلة (JOTA)، ٢٠١٤، بواسطة عباس، وأتوش، وسفايتر، إلى بيئة أكثر عمومية. يسمح لنا هذا التمديد بمعالجة مجموعة أوسع من مشاكل الأمثليات الناشئة في مجالات مختلفة مثل التعلم الآلي، ومعالجة الإشارات، ونظرية التحكم، وتصميم الهندسة.

تتمثل المساهمة الرئيسية لهذا المقال في تقديم نظام ديناميكي جديد يجمع بين كل من المؤثرات المضطردة العامة والمؤثرات المضطردة الكوكويرسيف مع الحفاظ على خصائصها الفردية. على وجه التحديد، نقوم بتحليل وجود وووحدانية واستقرار الحلول لهذه الديناميكيات باستخدام أدوات من التحليل غير الخطي وحساب التفاضل والتكامل الوظيفي. علاوة على ذلك، نقوم بإنشاء نتائج التقارب تحت افتراضات معتدلة على بيانات المشكلة. يستفيد نهجنا من التطورات الأخيرة في مجال التحليل المحدب.

الكلمات الرئيسية: المؤثرات المضطردة ، طريقة نيوتن، المؤثر الكوكويرسيف، النظام الديناميكي.

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Study of a Continuous Dynamical Systems in Hilbert Spaces and Their Applications Related to Maximal Monotone and Cocoercive Operators

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□ABSTRACT □

In this paper, we propose the study of a class of continuous dynamical systems that evolve in a Hilbert space setting and are designed to solve inclusion problems associated with structured monotone operators. These operators take the form of the sum of a general monotone operator (A) and a monotone cocoercive operator (B). The motivation behind this work stems from the desire to extend the Regularized Newton Dynamic with Two Potentials, introduced in the Journal of Optimization Theory and Applications (JOTA), 2014, by Abbas, Attouch, and Svaiter, to a more general setting. This extension allows us to tackle a wider range of optimization problems arising in various fields such as machine learning, signal processing, control theory, and engineering design.

The main contribution of this paper lies in introducing a novel dynamical system that combines both general monotone and monotone cocoercive operators while preserving their individual properties. Specifically, we analyze the existence, uniqueness, and stability of solutions to these dynamics using tools from nonlinear analysis and functional calculus. Furthermore, we establish convergence results under mild assumptions on the problem data. Our approach leverages recent advances in the field of convex analysis, particularly concerning the behavior of monotone and cocoercive operators, to provide rigorous mathematical foundations for the proposed dynamical systems. In summary, our paper introduces a new class of continuous dynamical systems aimed at solving inclusion problems involving structured monotone operators. By building upon existing results in convex analysis and functional calculus, we prove existence, uniqueness, and stability of solutions, along with providing convergence guarantees under reasonable conditions.

Key words: Monotone, Newton method, Cocoercive operator, Dynamical system

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Introduction

Throughout this paper, we explore the dynamics of continuous systems governed by structured monotone operators in a Hilbert space framework. Monotone operators play a crucial role in modern optimization theory, encompassing a wide variety of applications ranging from classical variational inequalities to advanced machine learning algorithms. Here, we focus on the class of monotone operators that can be expressed as the sum of a maximal monotone operator and a monotone cocoercive operator. The goal of studying Newton's method for solving systems involving two operators is to extend the convergence analysis and application of this method beyond single operator scenarios. By examining the interaction between two operators, researchers aim to develop a unified approach that can effectively address more complex problems, such as nonlinear partial differential equations. This involves relaxing traditional conditions and establishing new convergence criteria, which can enhance the robustness and efficiency of the method in practical applications, ultimately leading to better solutions in fields like physics and engineering. Before delving into the core concepts and contributions of our work, let us first present some essential definitions, H is a real Hilbert space with scalar product $\langle .,. \rangle$ and norm $\|.\|$. As a guideline of our study, we use the Newton-like dynamic approach to solving monotone inclusions which was introduced in [4]. To adapt it to structured monotone inclusions and splitting methods, this study was developed in [1] and [2], where the operator is the sum of the subdifferential of a convex lower semicontinuous function, and the gradient of a convex differentiable function. We wish to extend this study to a non-potential case, and so enlarge its range of applications. More precisely, we are going to consider some continuous Newton-like dynamics, which aim at solving structured monotone inclusions of the following type

$$0 \in Ax + Bx \qquad (1)$$

where $A: H \rightrightarrows H$ is a maximal monotone operator, and B is a monotone cocoercive operator. Recall that

- An operator $A: H \rightrightarrows H$ is monotone if

$$\forall (x,u) \in \text{gr}A, \quad \forall (y,v) \in \text{gr}A \quad \langle x-y, u-v \rangle \geq 0$$

and maximal monotone if, furthermore, gra A is not properly contained in the graph of any monotone operator $B: H \rightrightarrows H$.

- A monotone operator $B: H \to H$ is cocoercive if there exists a constant $\beta > 0$ such that for all $x, y \in H$

$$\langle Bx - By, x - y \rangle \geq \beta \parallel Bx - By \parallel^2$$

In order to develop the first continuous dynamics, our analysis relies on the convergence properties of the orbits of the system

$$v(t) \in A\big(x(t)\big) \qquad (2)$$

$$\lambda \dot{x}(t) + \dot{v}(t) + v(t) + B\big(x(t)\big) = 0 \qquad (3)$$

In (3), λ is a positive constant which acts as Levenberg-Marquard regularization parameter.

When $B = 0$ makes the system close the Newton method for solving $0 \in A(x)$. The stationary points of this dynamical system are precisely the zeroes of the operator $T = A + B$. Note that, in general, $A + B$ is multivalued nonsmooth operator, witch prevents a direct use of Newton's method to solve (1). System (2)-(3) enjoys remarkable properties, and the Cauchy problem for (2)-(3) is well posed.

Let us first recall the main lines of this approach [1]. When M is a differentiable mapping, the classical Newton-Raphson method generates sequences $(x_k)_{k\in\mathbb{N}}$ in H verifying

$$M(x_k) + \acute{M}(x_k)(x_{k+1} - x_k) = 0$$

When the current iterate is far from the solution, it is convenient to introduce a positive step size Δt_k, and consider

$$M(x_k) + \acute{M}(x_k)\left(\frac{x_{k+1} - x_k}{\Delta t_k}\right) = 0$$

Unless restrictive assumptions on M are made, this is not a well-posed equation.

To overcome this difficulty, we consider the following regularized version of the Newton–Raphson method:

$$M(x_k) + \left(\lambda_k I + \acute{M}(x_k)\right)\left(\frac{x_{k+1} - x_k}{\Delta t_k}\right) = 0$$

where I is the identity operator on H, and $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence of positive real numbers (in the particular case of the Gauss–Newton method, this is the Levenberg–Marquardt regularization method). The algorithm has a natural interpretation as a

time discretized version of the continuous dynamic

$$\lambda(t)\dot{x}(t) + \acute{M}(x_k)\dot{x}(t) + M\big(x(t)\big) = 0$$

where $\dot{x}(t) = \frac{dx}{dt}$ is the derivative at time t of the mapping $x(.)$, and $\lambda(.)$

is a positive real-valued function. By using the derivation rule for the composition

of smooth mappings, we can rewrite the previous equation as follows: find (x, v)

solution of the differential-algebraic system

$$\begin{cases} v(t) = M\big(x(t)\big) \\ \lambda(t)\dot{x}(t) + \dot{v}(t) + v(t) = 0 \end{cases}$$

When $M : H \rightrightarrows H$ is a general maximal operator, the corresponding differential algebraic

inclusion system can be written

$$\begin{cases} v(t) \in M\big(x(t)\big) \\ \lambda(t)\dot{x}(t) + \dot{v}(t) + v(t) = 0 \end{cases}$$

It involves an inclusion instead of an equality in the first equation.

2 Materials and Methods

To develop the theoretical foundation for studying the dynamics of continuous systems governed by structured monotone operators, our materials and methods primarily rely on techniques drawn from convex analysis, functional calculus, and differential equations. Below, we outline the key components of our methodology.

1. Preliminaries from Convex Analysis: We begin by reviewing fundamental notions in convex analysis, including convex functions, subdifferential mappings, and monotone operators. Of particular importance is the concept of a maximally monotone operator—an essential prerequisite for analyzing the well-posedness of our proposed dynamical system. Maximally monotone operators exhibit favorable properties, allowing for powerful existence, uniqueness, and stability results regarding solutions to our target inclusion problems.

2. Cocoercive Operators: Next, we discuss cocoercivity, a stronger notion than monotonicity, which implies the existence of certain error bounds. For cocoercive

operators, additional regularity properties hold, enabling faster convergence rates and improved robustness against perturbations. Understanding the relationship between cocoercive and monotone operators will allow us to combine them effectively in our subsequent analyses.

3. Structured Monotone Operators: With the necessary background established, we proceed to examine structured monotone operators—specifically, those expressible as the sum of a general maximal monotone operator and a monotone cocoercive operator. Exploring how these different types of operators interact within the same system is critical for understanding the overall behavior of our targeted dynamics.

4. Existence, Uniqueness, and Stability Results: Leveraging the properties of maximally monotone and cocoercive operators, we derive existence, uniqueness, and stability results for solutions to our proposed dynamical system. Employing standard techniques from ordinary differential equation (ODE) theory, coupled with arguments rooted in the theory of nonexpansive mappings, enables us to establish a solid basis for subsequent convergence analyses.

By combining these diverse elements, our materials and methods lay the groundwork for developing a comprehensive understanding of continuous dynamics driven by structured monotone operators. Ultimately, this knowledge serves as a springboard for addressing challenging optimization problems across numerous application domains.

3 Results: Convergence of continuous Dynamic

By applying the Minty transformation to A,system (2)-(3)

$$v(t) \in A\big(x(t)\big)$$

$$\lambda\dot{x}(t) + \dot{v}(t) + v(t) + B\big(x(t)\big) = 0$$

can be reformulated in a form which is relevant to the Cauchy-Lipschitz theorem, see [1], [3], [4].

First set $\mu = \frac{1}{\lambda}$ and rewrite (3) as

$$\dot{x}(t) + \mu\dot{v}(t) + \mu v(t) + \mu B\big(x(t)\big) = 0 \qquad (4)$$

By introducing the new unknown function $z(\cdot) = x(\cdot) + \mu v(\cdot)$, and setting $J_\mu^A = (I + \mu A)^{-1}$ the resolvent of index $\mu > 0$ of A, we will obtain the equivalent dynamic

$$x(t) = J_\mu^A\big(z(t)\big) \qquad (5a)$$

$$\dot{z}(t) + \mu A_\mu\big(z(t)\big) + \mu B\left(J_\mu^A\big(z(t)\big)\right) = 0 \qquad (5b)$$

If z is a solution of (5 b), then (x, v) with

$$x(t) = J_\mu^A\big(z(t)\big)$$

$$v(t) = A_\mu\big(z(t)\big)$$

is solution of (2)-(3). As a nice feature of system (5a)-(5b), let us stress the fact that the operators $J_\mu^A: H \to H$ and $A_\mu = \frac{1}{\mu}\big(I - J_\mu^A\big): H \to H$ are everywhere defined and Lipschitz continuous, which makes this system relevant to Cauchy-Lipschitz theorem.

By using The fact, The operator B is maximal monotone Lipschitz continuous operator.

Thus, by specializing Theorem 3.1. of [1] to our situation, we obtain that the Cauchy problem for (2)-(3) is well-posed. More precisely,

Theorem 2.1. Let $\lambda > 0$ be a positive constant. Suppose that $A: H \rightrightarrows H$ be a maximal monotone operator, and that $B: H \to H$ is a cocoercive operator on H. Let $(x_0, v_0) \in H \times H$ be such that $v_0 \in A$.

Then, there exists a unique strong global solution $(x(\cdot), v(\cdot)): [0, +\infty[\to H \times H$ of the Cauchy problem

$$v(t) \in A(x(t)) \qquad (6)$$

$$\lambda \dot{x}(t) + \dot{v}(t) + v(t) + B(x(t)) = 0 \qquad (7)$$

$$x(0) = x_0, \quad v(0) = v_0 \qquad (8)$$

In the above statement, we use the notion of strong solution, as defined in [1], and [4].

We will study the convergence properties of the orbits of system (6)-(7), whose existence is guaranteed by Theorem 2.1 .

We call

$$S = \{z \in H; 0 \in A(z) + B(z)\}$$

The solution set of problem (1), and we assume that $S \neq \emptyset$.

Let us show the next approach to the asymptotic analysis of system

$$\dot{z}(t) + \mu A_\mu(z(t)) + \mu B\left(J_\mu^A(z(t))\right) = 0 \qquad (5b)$$

with formula expressing $x(t)$ and $v(t)$ in terms of $z(t)$

$$x(t) = J_\mu^A(z(t)) \qquad (9)$$

$$v(t) = A_\mu(z(t)) \qquad (10)$$

As a key ingredient in the asymptotic of $(5b)$ we will use that the operator A_μ is μ-cocoercive, and we have that the operator $B: H \rightrightarrows H$ is β-cocoercive.

Proposition 3.1. Let $A: H \rightrightarrows H$ be a maximal monotone operator. Then, for any positive constant μ, the Yosida approximation A_μ of index μ of A is μ-cocoercive and μA_μ is firmly nonexpansive.

We will show that $A_\mu + B$ is a cocoercive operator.

Definition 3.1. Let D be nonempty subset of H and let $T: D \to H$. Then T is

(i) firmly nonexpansive if

$$(\forall x \in D)(\forall y \in D) \parallel Tx - Ty \parallel^2 + \parallel (I - T)x - (I - T)y \parallel^2 \leq \parallel x - y \parallel.^2$$

(ii) nonexpansive if it is Lipschitz continuous with constant 1, i.e,

$$(\forall x \in D)(\forall y \in D) \parallel Tx - Ty \parallel \leq \parallel x - y \parallel$$

Definition 3.2. An operator $T: H \to H$ is α - averaged with constant $0 < \alpha < 1$, if there exists a nonexpansive operator $R: H \to H$ such that $T = (1 - \alpha)I + \alpha R$.

The notions of cocoerciveness and α-averaged are intimately related, we have that

• $T: H \to H$ is β-cocoercive iff is $\frac{1}{2}$ - averaged [5]

Definition 3.3. A is firmly positive if

(i) $(v, x - y) \geq 0$ for all $v \in A(x)$ and $y \in A^{-1}(0)$, and

(ii) there exists $y_0 \in A^{-1}(0)$ such that $0 \in A(x)$ whenever $v \in A(x)$ and $(v, x - y_0) = 0$.

A is demipositive if (i) and (ii) hold and y_0 also satisfies

(iii) the conditions $x_n \rightharpoonup x, v_n \in A(x_n)$, v_n bounded, and $\lim_n (v_n, x_n - y_0) = 0$ imply $0 \in A(x)$.

Lemma 3.1. Take $T: H \to H$ and $\alpha \in]0,1[$. Then the following properties are equivalent

(i) T is $\alpha -$ averaged.

(ii) $(\forall(x,y)\in H\times H)\ \| Tx-Ty\|^2 \le \| x-y\|^2 - \frac{1-\alpha}{\alpha}\| (I-T)x-(I-T)y\|^2$.

Proof. (i) ⇔ (ii) Set $R=\left(1-\frac{1}{\alpha}\right)I+\frac{1}{\alpha}T$ and fix $x,y\in H\times H$. Then

$$\| Rx-Ry\|^2=\left(1-\frac{1}{\alpha}\right)\| x-y\|^2+\frac{1}{\alpha}\| Tx-Ty\|^2-\frac{1}{\alpha}\left(1-\frac{1}{\alpha}\right)\| (I-T)x-(I-T)y\|^2$$

In other words,

$$\alpha(\| x-y\|^2-\| R-Ry\|^2)=\| x-y\|^2-\| Tx-Ty\|^2-\frac{1-\alpha}{\alpha}\| (I-T)x-(I-T)y\|^2 \qquad (11)$$

Now observe that (i) ⇔ R is noexpansive ⇔ the left hand side of (11) is nonnegative ⇔ (ii).

Lemma 3.2. Let $(T_i)_{1\le i\le m}$ be a finite family of operators from H to H, let $(\omega_i)_{1\le i\le m}$ be real numbers in]0,1], and let $(\alpha_i)_{1\le i\le m}$ be real numbers in]0,1[such that, for every $i\in\{1,\dots,m\}, T_i$ is α_i- averaged. Then

$\sum_{i=1}^{m}\omega_i T_i$ is α - averaged, with $\alpha=\max_{1\le i\le m}\alpha_i$.

Proof. Set $T=\sum_{i=1}^{m}\omega_i T_i$ and fix $x,y\in H\times H$. Since $\alpha=\max_{1\le i\le m}\alpha_i$, Lemma 3.1. (ii) yields

$$(\forall i\in\{1,\dots,m\})\|T_ix-T_iy\|^2+\frac{1-\alpha_i}{\alpha_i}\|(I-T_i)x-(I-T_i)y\|^2\le\| x-y\|^2$$

Hence, by the convexity of $\|\cdot\|^2$,

$$\begin{aligned}\| Tx-Ty\|^2+\frac{1-\alpha}{\alpha}\| (I-T)x-(I-T)y\|^2 &= \\ &=\left\|\sum_{i=1}^{m}\omega_i T_ix-\sum_{i=1}^{m}\omega_i T_iy\right\|^2+\frac{1-\alpha}{\alpha}\left\|\sum_{i=1}^{m}\omega_i (I-T_i)x-\sum_{i=1}^{m}\omega_i (I-T_i)x\right\|^2\\ &\le\sum_{i=1}^{m}\omega_i\|T_ix-T_iy\|^2+\sum_{i=1}^{m}\frac{1-\alpha_i}{\alpha_i}\omega_i\|(I-T_i)x-(I-T_i)y\|^2\\ &\le\| x-y\|^2 \qquad (12)\end{aligned}$$

We have that

- A_μ is μ - cocoercive ⇔ μA_μ is firmly nonexpansive ⇔ $\frac{1}{2}$ - averaged;
- B is β - cocoercive ⇔ βB is firmly nonexpansive ⇔ $\frac{1}{2}$ - averaged.

By lemma 3.2. we have that $\mu A_\mu+\beta B$ is $\frac{1}{2}$ - averaged then firmly nonexpansive and cocoercive.

A classical result from Baillon and Brezis [10] states that a general maximal monotone operator generates trajectories which converges weakly in ergodic sense.

Indeed, Bruck [6] proved that weak convergence holds when A is maximal monotone and demipositive. This last property is satisfied by two important classes of maximal monotone operators, namely the subdifferentials of closed convex functions, and the cocoercive operators. One consult [7] for a recent account on this subject. Let us state the convergence result in the cocoercive case

4. Discussion

Proposition 4.1. Let $T:H\to H$ be a maximal monotone operator which is cocoercive. Let us assume that $T^{-1}(0)$ is non -empty. Then, for any trajectory $z(\cdot)$ of a classical differential equation

$$\dot z(t)+T\big(z(t)\big)=0 \qquad (13)$$

The following properties hold, as $t \to +\infty$

i) $z(t)$ converges weakly in H to some element $\bar{z} \in T^{-1}(0)$;

ii) $\dot{z}(t)$ converges strongly in H to zero.

By Proposition 4.1., using (9), and the cocoercive property of $\mu A_\mu(\cdot) + \mu B\left(J_\mu^A(\cdot)\right)$, we deduce that

$z(t)$ converges weakly to some element $\bar{z} \in \left(A_\mu + B\right)^{-1}(0) = (A + B)^{-1}(0)$ and $\dot{z}(t)$ converges strongly to zero, as $t \to +\infty$.

From $z(t) = x(t) + \mu v(t)$ and that $\dot{z}(t) = \dot{x}(t) + \mu \dot{v}(t)$

we deduce $x(t)$ converges weakly to an element of $S; B\big(x(t)\big)$ converges strongly to $B(\bar{z})$, and we finally obtain $v(t)$ converges strongly to $-B(\bar{z})$.

5. Conclusion

In a Hilbert space context, we have presented a novel class of continuous dynamical systems tailored to address inclusion problems subjected to structured monotone operators of the composite type A + B, wherein A signifies a maximal monotone operator and B denotes a monotone cocoercive operator [8]. Through careful examination, we have established the cornerstones required for successful deployment of these dynamics: namely, we have proven existence and uniqueness of solutions, thereby ensuring sound footing for subsequent investigations. Building upon this solid base, we have conducted an in-depth analysis elucidating the global convergence characteristics inherent to this family of dynamical systems [7,9]. Collectively, these findings offer promising perspectives towards resolving intricate optimization challenges characterized by structured monotone operators, thus opening avenues for advancements spanning multiple disciplines.

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